Mathematical Reviews

Edited by

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R. P. Boas, Jr., *Executive Editor*

Vol. 10, No. 5 May, 1949 pp. 277–346

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MATHEMATICAL REVIEWS

Published monthly, except August, by

THE AMERICAN MATHEMATICAL SOCIETY, Prince and Lemon Streets, Lancaster, Pennsylvania

Sponsored by

THE AMERICAN MATHEMATICAL SOCIETY
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Editorial Office

MATHEMATICAL REVIEWS, Brown University, Providence 12, R. I.

Subscriptions: Price $20 per year ($10 per year to members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions may be addressed to MATHEMATICAL REVIEWS, Lancaster, Pennsylvania, or Brown University, Providence 12, Rhode Island, but should preferably be addressed to the American Mathematical Society, 531 West 116th Street, New York 27, N. Y.

This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. These organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Entered as second-class matter February 3, 1940 at the post office at Lancaster, Pennsylvania, under the act of March 3, 1879. Accepted for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 538, P. L. and R. authorised November 9, 1940.

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Mathematical Reviews

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FOUNDATIONS

Götlind, Erik. On some equivalence theorems in two-valued logic. Norsk Mat. Tidsskr. **28**, 71–75 (1946). (Swedish)

Two proofs are given of the following theorem. Any statement built up by the biconditional and denial is tautologous if each component occurs an even number of times and there are an even number of denial signs. This theorem was stated without a proof by Quine [Mathematical Logic, Norton, New York, 1940, p. 60; these Rev. **2**, 65] who credits it to J. C. C. McKinsey. *B. Jónsson.*

***Kleene, S. C. On the intuitionistic logic.** Library of the Tenth International Congress of Philosophy, Amsterdam, August 11–18, 1948, Vol. I, Proceedings of the Congress, pp. 741–743 (1949).

A method is described whereby the unprovability of certain classically true formulas in the intuitionistic predicate calculus may be established by elementary means, in contrast to the method of Kleene [J. Symbolic Logic **10**, 109–124 (1945); these Rev. **7**, 406] and Nelson [Trans. Amer. Math. Soc. **61**, 307–368 (1947); these Rev. **10**, 3] which depends upon Gentzen's proof of consistency of intuitionistic arithmetic [Math. Ann. **112**, 493–565 (1936)]. The newer method applies Gentzen's theorem concerning a reduced form for proofs of predicate calculus formulas (the "Hauptsatz") [Math. Z. **39**, 176–210, 405–431 (1935)]. In addition to the examples of the earlier method, the formula $(x)(A \vee B(x)) \supset A \vee (x)B(x)$ has been shown to be unprovable. The author's detailed proofs are contained in a paper which has not yet been published. *D. Nelson.*

Griss, G.-F.-C. Logique des mathématiques intuitionistes sans négation. C. R. Acad. Sci. Paris **227**, 946–948 (1948).

The author sketches further aspects of his radical constructivism, rejecting the reductio ad absurdum as an intuitionistically meaningless procedure since a clear conception of an absurdity is held impossible [Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde **53**, 261–268 (1944); Nederl. Akad. Wetensch., Proc. **49**, 1127–1133 = Indagationes Math. **8**, 675–681 (1946); these Rev. **7**, 405; **8**, 307]. A mathematical statement is found to be meaningful only in case it is true. In the negationless propositional calculus, the only connectives to be admitted are conjunction and implication. An example shows that a disjunction may sometimes be interpreted as a statement about a union of sets. A portion of Heyting's intuitionistic propositional calculus [S.-B. Preuss. Akad. Wiss. Phys.-Math. Kl. **1930**, 42–56] may be interpreted as a calculus of sets. A null set, however, is not to be admitted on the ground that it can be defined only by a fictitious property. The author proposes to present at a later time a complete account of the negationless propositional calculus. *D. Nelson.*

Griss, G. F. C. Sur la négation (dans les mathématiques et la logique). Synthèse **7**, 71–74 (1948).

Britzelmayr, Wilhelm. Interpretation von Kalkülen. Synthèse **7**, 50–57 (1948).

Dürr, Karl. Logistik als Forschungsmethode. Synthèse **7**, 27–31 (1948).

Lectures at the Fourth International Significal Summer Conference.

Copilowish, Irving M. Matrix development of the calculus of relations. J. Symbolic Logic **13**, 193–203 (1948).

This is an expository article describing Peirce's method of correlating with every binary relation a matrix with values in a two-element Boolean algebra [Mem. Amer. Acad. Arts and Sciences (N.S.) **9**, 317–378 (1873)]. If we agree to consider a relation as a set of ordered pairs, then the corresponding matrix is essentially its characteristic function. *B. Jónsson* (Providence, R. I.).

Aubert, Karl Egil. On making precise and generalizing the concept of relation. Norsk Mat. Tidsskr. **30**, 33–53 (1948). (Norwegian)

Riabouchinsky, Dimitri. Sur les nombres d'origine imaginaire et la notion de signe d'un nombre complexe. C. R. Acad. Sci. Paris **225**, 1104–1106 (1947).

The author proposes the symbol "$\overline{|u}$" for "the number" s, if $u = |s|$. He rejects as in disaccord with the principle of Peacock-Hankel, of permanence of form, the use of $|x+iy|$ (x, y real), for $\sqrt{x^2+y^2}$, but accepts without inquiry "the identity $x^2 = |x|^2$." No mention is made of the exponential or logarithmic functions. *A. A. Bennett.*

Rossier, Paul. La géométrie et la théorie de la connaissance. Arch. Sci. Soc. Phys. Hist. Nat. Genève **1**, 460–486 (1948).

***Reichenbach, Hans. Philosophical foundations of probability.** Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability, 1945, 1946, pp. 1–20. University of California Press, Berkeley and Los Angeles, 1949. $7.50.

ALGEBRA

Dubisch, Roy. The number of r-tuples of pairs of integers. Amer. Math. Monthly **55**, 564–566 (1948).

The number of sets of pairs of integers 1 to n such that in any pair the first exceeds the second, and in any set the first integers are in rising order, the second in falling order, is shown to be $2^{n-1}-1$; the number with first pair (k, m) is $\binom{n-k+m-1}{n-2}$. *J. Riordan* (New York, N. Y.).

✱Sade, Albert. Énumération des carrés latins de côté 6. Marseille, 1948. 2 pp.

The divergent estimates of (reduced) Latin squares of side 6, 8192 by S. M. Jacob, and 9408 by Fisher and Yates, are brought to agreement by correcting an error in a combinatory formula used by the former, and the value 9408 is also obtained by an independent enumeration. This value, as well as the number 221 276 160 for squares of side 7, is said to have been obtained by Frolov [J. Math. Spéciales (3) 4, 25–30 (1890)] who gave without proof the following relation (I_n is the number of reduced Latin squares of side n, i.e., $L_n/n!(n-1)!$): $I_n/I_{n-1}=(I_{n-1}/I_{n-2})^2-I_{n-1}/2$, which is verified for $n=5$, 6 and by Frolov's number for 7.

J. Riordan (New York, N. Y.).

✱Sade, Albert. Enumération des Carrés Latins. Application au 7e Ordre. Conjecture pour les Ordres Supérieurs. Published by the Author, Marseille, 1948. 8 pp.

The author gives a complete tactical enumeration of the 7×7 Latin squares. The author's figure of 16 942 080 reduced 7×7 Latin squares exceeds Norton's [Ann. Eugenics 9, 269–307 (1939); these Rev. 1, 199] by 14112 but the author is able to explain this difference. The enumeration consists of an enumeration of the second and third lines together with a determination for each 3×7 rectangle of the number of Latin squares containing the three given top lines. It appears that the 146 species found by H. W. Norton comprise all species of 7×7 Latin squares, as was conjectured by Norton.

H. B. Mann (Columbus, Ohio).

Ullman, J. L. The number of distances in a cubical network. Amer. Math. Monthly 55, 562–563 (1948).

The generating function for distances between pairs of points on a cubical lattice of size n is found to be

$$A_0+2\sum_{d\geq 1} A_d x^{d^2}=[(n+1)+2\sum_{v=1}^{n}(n-v+1)x^{v^2}]^3.$$

J. Riordan (New York, N. Y.).

Müller, M. Ein Kriterium für das Nichtverschwinden von Determinanten. Math. Z. 51, 291–293 (1948).

The following theorem generalising several known facts [for references see the review of Massonnet, Bull. Soc. Roy. Sci. Liége 14, 313–317 (1945); these Rev. 8, 499] about nonvanishing determinants is proved: the determinant of an $n\times n$ matrix A differs from zero if and only if there is a matrix B such that the elements c_{ik} of the product $C=AB$ satisfy the inequalities: $|c_{ii}|>\sum_{i=1,i\neq k}^{n}|c_{ik}|$, $i=1, \cdots, n$.

O. Todd-Taussky (London).

MacDuffee, C. C. Orthogonal matrices in four-space. Canadian J. Math. 1, 69–72 (1949).

The author shows how any proper orthogonal 4×4 matrix may be expressed in the form

$$\begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix}\begin{bmatrix} b_0 & -b_1 & -b_2 & -b_3 \\ b_1 & b_0 & b_3 & -b_2 \\ b_2 & -b_3 & b_0 & b_1 \\ b_3 & b_2 & -b_1 & b_0 \end{bmatrix}.$$

He explains the connection with quaternions and with Wedderburn's expression for such a matrix as the exponential of a skew matrix [Ann. of Math. (2) 23, 122–134 (1921)].

H. S. M. Coxeter (New York, N. Y.).

Lee, H. C. Canonical factorization of non-negative Hermitian matrices. J. London Math. Soc. 23, 100–111 (1948).

A triangular matrix is said to be in canonical form if there exist positive integers $r_1<r_2<\cdots$ such that the kth column has zeros in the first r_k-1 rows while the element in the r_kth row is real and positive. It is proved that for every square matrix A there is a unitary matrix U such that $AU=T$ is triangular in canonical form, and that T is unique. If H is nonnegative Hermitian, there is a unique canonical T such that $H=TT^*$. Explicit formulae are obtained expressing the elements of T in terms of the elements of H.

C. C. MacDuffee (Madison, Wis.).

Castoldi, Luigi. "Forme aggiunte generalizzate" di una assegnata forma quadratica fondamentale. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 10(79), 135–141 (1946).

The νth adjoint of an n by n matrix A is the $\binom{n}{\nu}$-rowed matrix whose elements are the ν-rowed minor determinants of A with proper signs. Let A be of rank r and signature $S=P-N$, and let the adjoint be of rank R and signature S'. The author proves that $R=\binom{r}{\nu}$ for $r\geq\nu$. [See Niccoletti, Atti Accad. Sci. Torino 37, 655–659 (1902).] Also

$$S'=\binom{N}{0}\binom{P}{\nu}-\binom{N}{1}\binom{P}{\nu-1}+\binom{N}{2}\binom{P}{\nu-2}-\cdots.$$

C. C. MacDuffee (Madison, Wis.).

Dwyer, Paul S., and Macphail, M. S. Symbolic matrix derivatives. Ann. Math. Statistics 19, 517–534 (1948).

If Y is a matrix the elements y_{pq} of which are functions of the elements x_{mn} of another matrix X, then the four index quantity $\partial y_{pq}/\partial x_{mn}$ is regarded as a matrix for each fixed choice of m and n and also for each fixed choice of p and q. Systematic ways of obtaining these matrices are given when Y is a polynomial in X, its transpose and its inverse. The notation of tensor algebra is not employed, the purpose of the authors being to establish a matrix analogue of elementary differential calculus designed for application to statistics. The last third of the paper illustrates the use of the results in the theory of least squares, simultaneous regression, canonical correlation, orthogonal regression, components of total variance, the linear discriminant function, and in a theorem of multiple factor analysis.

W. Givens (Knoxville, Tenn.).

Blanuša, Danilo. Une manière de transformer une forme quadratique en somme de carrés. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 3, 1–5 (1948). (Croatian. French summary)

A minimum principle motivates the usual reduction of a quadratic form to a sum of squares. *C. C. MacDuffee.*

Blanuša, Danilo. Une démonstration des conditions pour qu'une forme quadratique soit définie. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 3, 6–10 (1948). (Croatian. French summary)

This is a proof by induction of the familiar conditions in terms of principal minors of its matrix that a quadratic form be positive definite. *C. C. MacDuffee.*

Abstract Algebra

Schützenberger, Marcel-Paul. Sur certaines applications remarquables des treillis dans eux-mêmes. C. R. Acad. Sci. Paris **227**, 1008–1010 (1948).

Given a complete lattice L, a mapping s of L into itself is called an "increasing S-mapping" if, for all x and y, (I) $x \leqq s(x)$; (II) $x \leqq y$ implies $s(x) \leqq s(y)$; (III) $s(x) \cap s(y) = s(x \cap y)$ [which imply $s(x) \cup s(y) \leqq s(x \cup y)$]. For each s there is a corresponding "decreasing S-mapping" t, where $t(x)$ is the intersection of all u such that $x \leqq s(u)$. The multiplicative system of these mappings is studied briefly. For each s there is defined an "S-relation"; namely, x and y are in this relation if $x \leqq y \leqq s(x)$. These S-relations are also characterized intrinsically. For a given L, the S-mappings form a lattice L^*, where $S \leqq S'$ if and only if $s(x) \leqq s'(x)$ for all x. It is stated that if L is distributive so is L^*, and that if L is complemented, then it is isomorphic to L^*.

P. M. Whitman (Silver Spring, Md.).

Schützenberger, Marcel-Paul. Sur l'extension des théorèmes de dualité aux treillis distributifs non complémentés. C. R. Acad. Sci. Paris **228**, 33–35 (1949).

The author continues the note reviewed above. In any modular lattice, σ denotes the "increasing S-mapping" in which $\sigma(x)$ is the join of the y such that y/x is complemented; τ is the dual of σ. In a distributive lattice with certain finiteness conditions, there is a unique x^* such that $x \cup x^* = \sigma(x)$ and $x \cap x^* = \tau(x^*)$; there is a dual relation $\dagger$, and $(x^*)\dagger = x = (x\dagger)^*$ but in general $x^{**} \neq x$. If x is meet-irreducible, then x^* is join-irreducible, and x and x^* principally split the lattice. The lattice is uniquely determined by the partially ordered set of join-irreducible elements (which is isomorphic to that of the meet-irreducible ones).

P. M. Whitman (Silver Spring, Md.).

Riguet, Jacques. Produit tensoriel d'ensembles ordonnés. C. R. Acad. Sci. Paris **227**, 1007–1008 (1948).

Given sets E_1 and E_2 with order relations Ω_1 and Ω_2, let Ω be the natural order relation in the direct product $E_1 \times E_2$; R the equivalence relation which identifies pairs in $E_1 \times E_2$ if either component is zero; $E^* = (E_1 \times E_2)/R$, $\Omega^* = \Omega/R$. The "tensor order product" of Ω_1 and Ω_2 is a certain extension of Ω^* to a complete lattice order. If E_1 and E_2 are complete lattices under their orders, then Ω^* modulo the distributive law gives the tensor product of two lattices, previously defined by the author [C. R. Acad. Sci. Paris **226**, 40–41, 143–146 (1948); these Rev. **9**, 265]. *P. M. Whitman.*

Kiss, S. A. Semilattices and a ternary operation in modular lattices. Bull. Amer. Math. Soc. **54**, 1176–1179 (1948).

In a modular lattice, the author defines the ternary operation $[x, t, y] = \{x \cap (t \cup y)\} \cup (t \cap y) = \{x \cup (t \cap y)\} \cap (t \cup y)$, generalizing the operation of Kiss and Birkhoff [same Bull. **53**, 749–752 (1947); these Rev. **9**, 76] for distributive lattices and Grau [same Bull. **53**, 567–572 (1947); these Rev. **9**, 3] for Boolean algebras. Then $[x, t, x] = x$, and $[[x, t, y], t, z] = [x, t, [y, t, z]]$. For given t, those ternary products for which $[x, t, y] = [y, t, x]$ form a semilattice (greatest lower bounds exist, but not always least upper). It is conjectured that covering relations in a modular lattice are preserved in this derived semilattice. If $[x, t, x'] = t$ for all t, x and x' are called strictly complementary; strict complementation is unique. We have $[0, t, I] = t$; $[x, 0, y] = x \cap y$; $[x, I, y] = x \cup y$. *P. M. Whitman* (Silver Spring, Md.).

Nakayama, Tadasi. Remark on direct product decompositions of a partially ordered system. Math. Japonicae **1**, 49–50 (1948).

Generalizing existing results [for example, G. Birkhoff, Duke Math. J. **9**, 283–302 (1942); these Rev. **4**, 74], the author proves that two finite direct product decompositions of a doubly directed system have always a common refinement. Further, if α is an infinite cardinal, two direct product decompositions of an α-complete doubly directed system into at most α such systems have always a common refinement. *P. M. Whitman* (Silver Spring, Md.).

Hsu, Chen-Jung. On lattice theoretic characterization of the parallelism in affine geometry. Ann. of Math. (2) **50**, 1–7 (1949).

The author develops a theory of parallelism without using chain conditions. One writes $(b, c)M$ if $a \cup (b \cap c) = (a \cup b) \cap c$ for all $a \leqq c$. In a (relatively) complemented semi-modular lattice which is irreducible (in the sense that no element except 0 and I has a unique complement), $b \| c$ means $b' \cup c = b \cup c' = b \cup c$ for every b' and c' such that $0 < b' \leqq b$ and $0 < c' \leqq c$, while $b \Pi c$ means that $b \| c$ and also $(b \cup c) \cap b' = b$ and $(b \cup c) \cap c' = c$ for every b' and c' such that $b \leqq b'$, $c \leqq c'$, $b' \cap c = b \cap c' = 0$; b and c are usually assumed distinct. Then $\|$ and Π are reflexive and symmetric. If $b \| c$ and $b \leqq b'$ then $b' \cap c = 0$ or $c < b'$; if both the latter hold, then (b', c)non-M. The above results also hold for Π. If b and c cover 0 then $b \Pi c$. If $b \| c$ and there exist p and q covering 0 such that $p \leqq b$, $q \leqq c$, then $b \Pi c$. If $b \Pi c$, then $b \cup c$ covers c. An affine lattice is an irreducible complemented semi-modular lattice in which: if $b' \cup c = b \cup c$ for all b' such that $0 < b' \leqq b$, then there exists a unique b' such that $b \leqq b'$ and either $b' \Pi c$ or $b' \| b$ according as the author considers Π or $\|$. In such a lattice, Π and $\|$ are transitive. Also, if $a \Pi a'$, $b \Pi b'$, $a \cap b \neq 0$, and $a' \cap b' \neq 0$, then $a \cup b \Pi a' \cup b'$ and $a \cap b \Pi a' \cap b'$, the first of these two conclusions holding also for $\|$.

P. M. Whitman (Silver Spring, Md.).

Birkhoff, Garrett, and Frink, Orrin, Jr. Representations of lattices by sets. Trans. Amer. Math. Soc. **64**, 299–316 (1948).

In the first part necessary and sufficient conditions are given for an abstract lattice L to be isomorphic with the lattice of all subalgebras of a suitable abstract algebra with finitary operations. These conditions consist in the property that L is complete, every element of L is a join of $\uparrow$-inaccessible elements and further that $X_\alpha \uparrow X$, $Y_\beta \uparrow Y$ imply $(X_\alpha \cap Y_\beta) \uparrow X \cap Y$ for any directed sets. It is shown that on these conditions every element of L is a meet of completely meet-irreducible elements.

In the second part representations of lattices by sets and in particular by sets of dual ideals are studied. A dual ideal of a lattice L is a set D of elements of L such that D contains the meet $x \cap y$ of two elements of L if and only if it contains both x and y. A meet-representation of L is a correspondence $x \rightarrow R(x)$ between the elements $x \epsilon L$ and certain sets $R(x)$, such that $R(x \cap y)$ is the set-intersection of $R(x)$ and $R(y)$. It is shown that in considering meet-representations by sets of objects of any kind there is no loss of generality if one confines attention to sets of dual ideals. The authors' main results concerning meet-representations show that meet-representations correspond to sets of dual ideals, and that isomorphic representations are given by all meet-irreducible, by all completely meet-irreducible, and by all principal dual ideals. *O. Borůvka* (Brno).

Sikorski, R. On a generalization of theorems of Banach and Cantor-Bernstein. Colloquium Math. 1, 140–144 (1948).

If $\mathbf{A}$ is a Boolean algebra and $E\epsilon\mathbf{A}$, then the author designates by $E\mathbf{A}$ the set of all elements $A\epsilon\mathbf{A}$ such that $A\subset E$. As usual, A' denotes the complement of A. The author proves the following generalizations, to Boolean algebras, of a theorem of S. Banach [Fund. Math. 6, 236–239 (1924)] and of F. Bernstein's equivalence theorem. Let $\mathbf{A}$ and $\mathbf{B}$ be two σ-complete Boolean algebras, let $A\epsilon\mathbf{A}$ and $B\epsilon\mathbf{B}$. (1) If f and g are two σ-homomorphisms of $\mathbf{A}$ in $B\mathbf{B}$ and of $\mathbf{B}$ in $A\mathbf{A}$, respectively, then there exist two elements $C\epsilon\mathbf{A}$ and $D\epsilon\mathbf{B}$ such that $f(C)=D'$ and $g(D)=C'$. (2) If $\mathbf{A}$ is isomorphic to $B\mathbf{B}$ and $\mathbf{B}$ to $A\mathbf{A}$, then $\mathbf{A}$ and $\mathbf{B}$ are isomorphic.

A. Rosenthal (Lafayette, Ind.).

Brauer, Richard. A note on Hilbert's Nullstellensatz. Bull. Amer. Math. Soc. 54, 894–896 (1948).

Verf. beweist den Hilbertschen Nullstellensatz ohne Benutzung der Idealtheorie. Die algebraischen Mannigfaltigkeiten $f_1=0, \cdots, f_r=0$ werden nach der Anzahl der auftretenden Variablen und einer Gradzahl teilweise geordnet. Der Beweis wird dann durch Induktion nach dieser Ordnung geführt.

P. Lorenzen (Cambridge, England).

Fuchs, L. A condition under which an irreducible ideal is primary. Quart. J. Math., Oxford Ser. 19, 235–237 (1948).

In a commutative ring R an irreducible ideal Q is primary if and only if for every b in R the chain $Q\subseteq Q:(b)\subseteq Q:(b^2)\subseteq\cdots$ is finite.

I. S. Cohen (Cambridge, Mass.).

Dieudonné, Jean. Les semi-dérivations dans les extensions radicielles. C. R. Acad. Sci. Paris 227, 1319–1320 (1948).

Dieudonné, Jean. Théorie de Galois des extensions radicielles d'exposant quelconque. C. R. Acad. Sci. Paris 228, 148–150 (1949).

Let K be a field of characteristic $p>0$. If D is an endomorphism of the additive group of K, N_D denotes the set of all $x\epsilon K$ such that $D(xy)=xD(y)$ for all $y\epsilon K$, and S_D denotes the set of all $x\epsilon K$ such that $D(xy)=xD(y)+(Dx)y$ for all $y\epsilon K$. Then D is a semi-derivation of K of height r if $K^{p^r}\subset S_D$. The author obtains results of which the following is a special case. If $\mathbf{L}_r$ denotes the set of all subfields L of K such that $K^{p^{r-1}}\subset L\supset K^{p^r}$, $[K:L]<\infty$, then for each $L\epsilon\mathbf{L}_r$ the set Δ_L of all semi-derivations D of K of height r such that $L\subset N_D\cap S_D$, $D(L(K^{p^r}))\subset L(K^{p^r})$ is a Lie ring (for the operations of addition and commutation) which satisfies a certain list of conditions; conversely, if $\mathbf{\Delta}_r$ is the set of all Lie rings Δ of semi-derivations of K of height r satisfying this list of conditions then for each $\Delta\epsilon\mathbf{\Delta}_r$ the field $L_\Delta=\bigcap N_D$ over $D\epsilon\Delta$ belongs to $\mathbf{L}_r$; $L\rightarrow\Delta_L$ is a one-to-one mapping of $\mathbf{L}_r$ onto $\mathbf{\Delta}_r$ for which $\Delta\rightarrow L_\Delta$ is the inverse mapping. For $r=0$ this specializes to N. Jacobson's Galois theory of purely inseparable fields of exponent 1 [Trans. Amer. Math. Soc. 42, 206–224 (1937); Amer. J. Math. 66, 645–648 (1944); these Rev. 6, 115]. Footnote (1) of the second paper is in error and should refer, instead, to the first paper.

E. R. Kolchin (New York, N. Y.).

Kaplansky, Irving. Polynomials in topological fields. Bull. Amer. Math. Soc. 54, 909–916 (1948).

Extensions to topological fields of the following theorem of Habicht [Comment. Math. Helv. 18, 331–348 (1946); these Rev. 8, 61] are studied: a polynomial in n variables over a real closed field (in the sense of Artin and Schreier) which is positive for $-m\leqq x_i\leqq m$ has a positive lower bound in this region. In the generalisation an abstract definition of boundedness is essential. The fields are assumed of type V, i.e., the inverses of the elements of any set bounded away from O are bounded. Using an earlier theorem of the author [Duke Math. J. 14, 527–541 (1947); these Rev. 9, 172], restated in the language of filters, the following generalisation of a well-known fact is proved: a polynomial in one variable in a field which is algebraically closed in its completion maps closed sets into closed sets. For $n\geqq2$ conditions on the field and on sets S (in the n-dimensional vector space over F) are found which ensure that a polynomial maps S into a set bounded away from O. In particular, these results imply that Habicht's theorem is true even when instead of "polynomial" we have "rational function." The case $n\geqq3$ appears more difficult and is treated differently.

O. Todd-Taussky (London).

Chevalley, Claude. Sur la classification des algèbres de Lie simples et de leurs représentations. C. R. Acad. Sci. Paris 227, 1136–1138 (1948).

Let L be a simple Lie algebra over an algebraically closed field. By examining all the cases separately, Cartan showed that the algebras L are in one-to-one correspondence with certain systems of integers S. There remained the problem of supplying an algebraic proof a priori of the existence of an L for given S. The present note contains a sketch of such a proof. The author at the same time settles the question of representations. Associated to a system S of rank l are certain weights (linear forms in l variables) among which can be distinguished certain dominant weights. To each irreducible representation of the algebra associated with S corresponds a maximal dominant weight which determines the representation up to an equivalence. By examining the different systems S separately, Cartan showed that each dominant weight is the maximal dominant weight of an irreducible representation. Weyl proved this fact a priori by transcendental methods. The present note sketches an algebraic proof.

P. A. Smith (New York, N. Y.).

Chevalley, Claude. Sur les représentations des algèbres de Lie simples. C. R. Acad. Sci. Paris 227, 1197 (1948).

It is remarked that a different proof of the results referred to in the preceding review has been obtained simultaneously and independently by Harish-Chandra.

P. A. Smith.

THEORY OF GROUPS

✱Rutherford, Daniel Edwin. Substitutional Analysis. Edinburgh, at the University Press, 1948. xi+103 pp. 25s.

The author states "the purpose of this book is to give an account of the methods employed by Alfred Young in his reduction of the symmetric group and to describe the more important results achieved by him." Although the theory itself possessed much natural elegance many of Young's original proofs were quite intricate. By rearranging the order of presentation, by use of some results of von Neumann and some results of the reviewer, and by introduction of a considerable number of his own new proofs the author has arrived at a treatment which is thoroughly elegant in method and theory. Moreover, the treatment is quite elementary:

all that is required of the reader is a little knowledge of matrices and finite groups.

The first chapter is introductory. The second chapter takes up the calculus of tableaux. Let $\alpha_1+\cdots+\alpha_k=n$, $\alpha_1\geqq\cdots\geqq\alpha_k>0$, be a partition of n. With this partition is associated a diagram or shape consisting of rows and columns of spaces (squares), α_1 spaces in the first row, $\cdots$, α_k spaces in the kth row, the rth space of any row lying in the rth column of the shape. An arrangement of the "letters" $1, \cdots, n$ in the spaces of the shape belonging to a partition is called a tableau. For any tableau T denote by P the sum of all permutations of the letters $1, \cdots, n$ which leave fixed the totality of letters in each row of T. Denote by N the alternating sum of all permutations which leave fixed the elements of each column of T, the plus sign being chosen for even and the minus sign for odd permutations. The main result of this chapter is the formula $PNPN=\theta PN$, where $\theta=n!/f$, f being the degree of the irreducible representation of the symmetric group S_n associated with the partition to which the tableau T belongs. The expression $E=PN/\theta$ is a primitive idempotent in the group algebra Γ_n of S_n. In the third chapter these primitive idempotents in Γ_n are used to define certain other idempotents and elements of Γ_n called respectively the semi-normal idempotents and semi-normal units, in terms of which a complete set of irreducible representations of S_n, called the semi-normal representations, can be written down explicitly. In chapter IV two other sets of irreducible representations are given; of these one, the normal or orthogonal representations, are very closely related to the semi-normal representations and are as the name suggests orthogonal; the other set called the natural representations represent each group element by matrices having integral entries. The last two chapters deal with characters and substitutional equations.

A good many pages of the book are devoted to proving for the symmetric group theorems that hold for all finite groups and which are classical parts of the theory of representations of finite groups. The reviewer regrets that such topics as the alternating group and the full linear group, to which Young's theory applies quite readily, were not included. *R. M. Thrall* (Ann Arbor, Mich.).

***Piccard, Sophie. Sur les Bases du Groupe Symétrique. II.** Librairie Vuibert, Paris, 1948. 119 pp.

[For Part I see Mém. Univ. Neuchâtel, vol. 19 (1946); these Rev. 8, 13.] The author's introduction is as follows. Soit $\mathfrak{S}_n(\mathfrak{A}_n)$ le groupe symétrique (alterné) des substitutions des éléments $1, 2, \cdots, n$ (n=entier $\geqq 4$). On appelle base du groupe $\mathfrak{S}_n(\mathfrak{A}_n)$ tout couple de substitutions de $\mathfrak{S}_n(\mathfrak{A}_n)$, génératrices de ce groupe. Le présent ourvage se compose de deux parties. Dans la première partie, nous établissons des critères permettant de determiner toutes les bases des groupes $\mathfrak{S}_n$ et $\mathfrak{A}_n$, dont l'une des substitutions se compose de deux transpositions. La seconde partie est consacrée à la recherche de toutes les bases du groupe $\mathfrak{S}_7$, dont nous indiquons les types et un système complet de représentants indépendants. En fin de l'ouvrage, nous indiquons le sommaire des critères permettant de discerner toutes les bases des groupes $\mathfrak{S}_3$, $\mathfrak{S}_4$, $\mathfrak{S}_5$ et $\mathfrak{S}_6$. *G. de B. Robinson.*

Kochendörffer, Rudolf. Über treue irreduzible Darstellungen endlicher Gruppen. Math. Nachr. 1, 25–39 (1948).

With any representation Γ of a finite group G is associated the normal subgroup $K(\Gamma)$ consisting of those elements of G which map into identity under Γ. The author investigates conditions on G for the existence of a faithful irreducible representation Γ of G, i.e., one with $K(\Gamma)=1$. An isomorphism between any two subgroups of G is called coherent if it commutes with the inner automorphisms of G. Let P be the group generated by all Abelian minimal normal subgroups of G; then P can be written as a direct product $P=J_1\times\cdots\times J_t$, where J_i is, for each i, a direct product $J_i=F_{i1}\times\cdots\times F_{in_i}$ of minimal invariant subgroups of G such that F_{ij} is coherently isomorphic to F_{hk} if and only if $i=h$. Suppose that F_{i1} has order p^{r_i} and has g_i' coherent automorphisms. Then $g_i'+1=p^{s_i}$. The main result is that G has a faithful irreducible representation Γ over the complex field if and only if $g_i s_i\leqq r_i$, for all primes p, for every p-coherent J_i of G. Several consequences of this theorem are given and similar results are obtained for fields of characteristic p and for collineation representations. The author seems to have overlooked a paper by L. Weisner [Amer. J. Math. **61**, 709–712 (1939); these Rev. 1, 6] which overlaps the present work. *R. M. Thrall* (Ann Arbor, Mich.).

Lombardo-Radice, Lucio. Il difetto di regolarità di un gruppo finito rispetto a un divisore primo del suo ordine. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 7, 169–183 (1948).

Let G be a group whose order is divisible by the prime p, A the group algebra of G over $GF(p)$, C the centre of A and h_p the number of classes of conjugate elements of G whose orders are divisible by p. The author proves that the radical of C has rank at least h_p by actually exhibiting this number of properly nilpotent independent centre elements. He defines $\delta=e-h_p$, where e is the rank of the radical of C, to be the "defect of regularity" (difetto di regolarità) of G, and remarks that δ may also be defined as the difference between the number of nonequivalent irreducible modular representations of G and the number of primitive idempotents of C. These results are immediate consequences of the Brauer-Nesbitt modular representation theory [see, for example, Ann. of Math. (2) **42**, 556–590 (1941); these Rev. 2, 309]. *S. A. Jennings* (Vancouver, B. C.).

Seki, Takejiro. Über die Existenz der Zerfällungsgruppe in der Erweiterungstheorie der Gruppen. Tôhoku Math. J. 48, 235–238 (1941).

Let $\mathfrak{G}$ be a group which includes $\mathfrak{N}$ as a normal divisor, and let $\mathfrak{G}/\mathfrak{N}\cong\mathfrak{F}$. The author shows that $\mathfrak{G}$ has a splitting group ("Zerfällungsgruppe") $\bar{\bar{\mathfrak{G}}}$ [Zassenhaus, Lehrbuch der Gruppentheorie I, Teubner, Leipzig and Berlin, 1937, p. 98]. This theorem is a generalization of the result of Artin [Zassenhaus, loc. cit.] where $\mathfrak{N}$ is Abelian. Let $\{S_\sigma\}$, $\sigma\in\mathfrak{F}$, be a set of representatives of the cosets of $\mathfrak{G}$ modulo $\mathfrak{N}$. Then $\bar{\bar{\mathfrak{G}}}$ includes a normal divisor $\bar{\bar{\mathfrak{N}}}$ for which $\mathfrak{G}=\sum_{\sigma\in\mathfrak{F}}\mathfrak{N}S_\sigma$, $\bar{\bar{\mathfrak{G}}}=\sum_{\sigma\in\mathfrak{F}}\bar{\bar{\mathfrak{N}}}S_\sigma$ and $\bar{\bar{\mathfrak{N}}}\supset\mathfrak{N}$. Then, $\bar{\bar{\mathfrak{G}}}=\bar{\bar{\mathfrak{N}}}\times\bar{\bar{\mathfrak{F}}}$, a direct product, where $\bar{\bar{\mathfrak{F}}}\cong\mathfrak{F}$. Also, $\bar{\bar{\mathfrak{G}}}=\mathfrak{G}\bar{\bar{\mathfrak{F}}}$; and if $\mathfrak{F}$ is Abelian, this last product is direct. *F. Haimo* (St. Louis, Mo.).

Polya, Georges, et Meyer, Burnett. Sur les symétries des fonctions sphériques de Laplace. C. R. Acad. Sci. Paris 228, 28–30 (1949).

The authors describe a method for classifying the finite groups of congruent transformations in ordinary space. For each group they state, without proof, the form of the generating function $F(t)=h_0+h_1t+h_2t^2+\cdots$ for the number h_n of linearly independent harmonic polynomials of degree n, invariant under the group. For instance,

the group generated by reflections in three planes inclined at angles $\pi/2$, π/p, π/q has $F(t)=1/(1-t^h)(1-t^{g/2h})$, where $g=8pq/[4-(p-2)(q-2)]$ and

$$\cos^2(\pi/h)=\cos^2(\pi/p)+\cos^2(\pi/q)$$

[see Coxeter, Regular Polytopes, Methuen, London, 1948; Pitman, New York, 1949, pp. 19, 82; these Rev. **10**, 261]. Again, the subgroup of order $g/2$ consisting of rotations alone has $F(t)=(1+t^{h-1+g/2h})/(1-t^h)(1-t^{g/2h})$. Incidentally, the exponent $h-1+g/2h$ is equal to the number of reflections occurring in the group generated by reflections.

H. S. M. Coxeter (New York, N. Y.).

Vilenkin, N. Ya. On the classification of zero-dimensional locally compact Abelian groups. Doklady Akad. Nauk SSSR (N.S.) **61**, 969–971 (1948). (Russian)

This note states a sequence of theorems which give the complete structural analysis of a class of locally compact zero-dimensional groups of type P. For notations and definitions one is referred to an earlier note [Rec. Math. [Mat. Sbornik] N.S. **19**(61), 311–340 (1946); see also ibid., 85–154 (1946); these Rev. **8**, 312, 132]. The class of groups is restricted by conditions which are generalizations of the concepts of regular stratification and regular subgroup used previously by the author. The restrictions imposed on G of type P and on a certain compact subgroup H, both open and closed in G, are that for every integer n and a: (1) ${}_n[\overline{p^aG}]=\overline{{}_n[p^aG]}$, (2) $p^nG\cap\overline{p^{a+n}G}=p^n(\overline{p^aG})$, (3) $p^nG\cap G_a=p^nG_a$, (4) $p^nH\cap p^{a+n}G=p^n(H\cap p^aG)$, (5) ${}_\omega[G/\overline{{}_\omega[G]}]=0$. As stated by the author, the theorems of this paper constitute the analogues of the now well-known analysis of the discrete primary countable groups.

L. Zippin (Flushing, N. Y.).

Gel'fand, I. M., and Naĭmark, M. A. On the connection between the representations of a complex semi-simple Lie group and those of its maximal compact subgroups. Doklady Akad. Nauk SSSR (N.S.) **63**, 225–228 (1948). (Russian)

The "nondegenerate" continuous (in the strong topology) irreducible unitary representations on Hilbert spaces of complex semi-simple Lie groups (especially of the complex unimodular group), and their contractions to maximal compact subgroups, are described in concise terms, in continuation of earlier work by the same authors [Mat. Sbornik N.S. **21**(63), 405–434 (1947); Izvestiya Akad. Nauk SSSR. Ser. Mat. **11**, 411–504 (1947); these Rev. **9**, 328, 495]. Any such representation $a\rightarrow T_a$ of the group G has as a representation space a Hilbert space of all square-integrable functions, relative to a measure depending on the representation (though simply the unique invariant measure for the representations in the principal series), over the right coset space U/Γ, where U is a maximal compact subgroup of G and $\Gamma=U\cap D$, D being a maximal Abelian subgroup of G generated by a regular element. Each coset of U modulo Γ is contained in exactly one right coset of G modulo K, where K is the subgroup of G generated by the positive roots of its Lie algebra. The functions f on U/Γ can thereby be identified with those functions $\bar{f}$ on G/K which have the property $\bar{f}(\gamma u)=\bar{f}(u)$, $\gamma\epsilon\Gamma$, and the integral of f over U/Γ is the same as the integral of $\bar{f}$ over G/K (relative to the respective invariant measures). The representation T can be most conveniently described by its action on the Γ-invariant functions f over G/K, and has then the form $(T_af)(x)=\alpha(xa)(\alpha(x))^{-1}f(xa)$, where α is a function determined (via a way of factoring the elements of G) by a character χ of D (of absolute value one for the principal series and not necessarily bounded for the complementary series), and is uniquely determined by the equivalence class of the representation within multiplication by a function of absolute value one. In case G is the complex unimodular group, U can be taken as the unitary elements, D as the diagonal elements, and K as those elements which are zero below the diagonal.

A necessary and sufficient condition that T have in its representation space a nonzero vector x invariant under the T_a, $a\epsilon U$, is that χ be trivial on Γ; if x exists, it is unique within multiplication by nonzero numbers, and (T_ax, x) is a positive definite function on G which is invariant under two-sided translations by elements of U. This function is called the spherical function of the given representation and an explicit formula is given for it, in terms of χ. A necessary and sufficient condition that the contraction of T to U contain a given continuous irreducible unitary representation S of U is that the representation space of S contain a weight vector for the contraction of χ to Γ; and the maximum number of linearly independent weight vectors is the same as the number of times S is contained in T. In particular, the representation of U corresponding to the lowest dominant weight occurs only once, and that weight is the contraction of χ to Γ.

I. E. Segal (Chicago, Ill.).

Monna, A. F. Sur un théorème de M. J. F. Koksma concernant la théorie des approximations diophantiques. Nederl. Akad. Wetensch., Proc. **51**, 457–469 = Indagationes Math. **10**, 151–163 (1948).

L'auteur prouve le théorème suivant qui est une généralisation d'un théorème de Koksma [mêmes Proc. **43**, 211–214 (1940); ces Rev. **1**, 202]. Soit G un groupe (multiplicatif) dans lequel une mesure $m(A)$ invariante à gauche est définie [c'est-à-dire $m(\alpha A)=m(A)$ pour $\alpha\epsilon G$, $A\subset G$]. Soit $F(x, y)\epsilon G$ une fonction définie pour $x\epsilon N$ [N étant un ensemble dénombrable] et pour $y\epsilon Y$ [Y étant un ensemble arbitraire]. Soit $Q\subset G$ et soient $V(x)$ pour $x\epsilon N$ des sous-ensembles mesurables de Q. Supposons que, pour chaque $x\epsilon N$, il existe au plus $K(x)<\infty$ points $y\epsilon Y$ tels que $F(x, y)$ soit un point de $Q^2=Q\cdot Q$. Soit $\sum_{x\epsilon N}K(x)m(V^{-1}(x))$ convergent. Alors pour presque tous les points α de Q il existe au plus un nombre fini de couples (x, y) pour lesquels la relation $\alpha^{-1}F(x, y)\epsilon V(x)$ est satisfaite. [La condition (6), p. 460, introduite par l'auteur est superflue. Dans l'application de ce théorème au groupe multiplicatif dans le corps des nombres réels l'auteur n'a pas fait attention à la circonstance que la mesure habituelle dans ce corps n'est pas invariante par rapport à la multiplication.]

V. Knichal (Prague).

NUMBER THEORY

Fitting, F. Panmagische 64 feldrige Quadrate mit bimagischen Diagonalen. Tôhoku Math. J. **48**, 239–244 (1941).

H. Schots [Acad. Roy. Belgique. Bull. Cl. Sci. (5) **17**, 339–361 (1931), p. 357] constructed a doubly magic pandiagonal square of order 8 having four trebly magic diagonals, and later [ibid. **25**, 118–124 (1939)] a pandiagonal square that is doubly magic for all its 16 diagonals (including the broken diagonals) though not for rows nor columns. This

means that the sum of the squares of the 8 numbers in each of the 16 diagonals is 11180. The author of the present note expresses the 64 numbers in the binary scale, and by permuting the digits derives a family of $2^6\cdot 6!=46080$ such pandiagonal squares. *H. S. M. Coxeter.*

Stern, Erich H. Carrés magiques avec certains nombres donnés à des places choisies. Revista Mat. Timişoara **23**, 7 pp. (1943).

The chief result is that a magic square of order 6 or 8 can be constructed when four of the numbers are in prescribed positions, provided not more than two of these positions are in the same row, column, or diagonal. *H. S. M. Coxeter.*

Palamà, G. Sul problema di Escott-Tarry. Boll. Mat. (5) **2**, 25–26 (1948).

This paper gives the previously known ideal solutions of the Tarry-Escott problem [see Gloden, Mehrgradige Gleichungen, Noordhoff, Groningen, 1944; these Rev. **8**, 441] for degrees less than 10. For $n=10(1)25$ the author gives the smallest known number of integers in each of the two sets which have equal sums of like powers up to and including the nth powers. For n beyond 15 these numbers of integers are considerably greater than the ideal $n+1$.
D. H. Lehmer (Berkeley, Calif.).

Mirsky, L. A remark on D. H. Lehmer's solution of the Tarry-Escott problem. Scripta Math. **14**, 126–127 (1948).

An alternative proof is given of a theorem of Lehmer having to do with equal sums of like powers of integers [Scripta Math. **13**, 37–41 (1947); these Rev. **9**, 78]. Instead of using a generating function, the author makes a simple application of the multinomial theorem.
D. H. Lehmer (Berkeley, Calif.).

Sprague, R. Über Zerlegungen in ungleiche Quadratzahlen. Math. Z. **51**, 289–290 (1948).

The author shows in a quite elementary way that every positive integer greater than 128 is the sum of distinct squares. The proof depends on showing with a small list of such representations that the integers between 129 and 256 are sums of distinct squares not exceeding 100. There are exactly 32 integers $2, 3, 6, 7, 8, 11, \cdots, 128$ which are not the sum of distinct squares. *D. H. Lehmer.*

Ward, Morgan. The law of repetition of primes in an elliptic divisibility sequence. Duke Math. J. **15**, 941–946 (1948).

The main result is a theorem concerning the rank of apparition of p^n, p a prime, in an elliptic sequence. This theorem is an analogue of the law of repetition for Lucas sequences and it allows the author to prove a result previously announced [Amer. J. Math. **70**, 31–74 (1948); these Rev. **9**, 332]. *H. S. Zuckerman* (Seattle, Wash.).

Ward, Morgan. Euler's problem on sums of three fourth powers. Duke Math. J. **15**, 827–837 (1948).

The famous problem referred to in the title is that of proving or disproving that a biquadrate is not the sum of three biquadrates. This paper proves that the Diophantine relation

$$(1)\qquad x^4+y^4+z^4=w^4, \qquad xyw\neq 0,$$

implies (2) $w>10^4$. The basic idea of the proof is set forth in a previous paper [Proc. Nat. Acad. Sci. U. S. A. **31**, 125–127 (1945); these Rev. **6**, 259]. In the present paper a full account of the details is given. The assumption that (1) has a solution leads to the equation

$$(3)\qquad u^4+v^4=2^s kl(2^{8w+2s+16}d^8k^2+e^8l^2).$$

The denial of (2) gives corresponding restrictions on the variables on the right side of (3). All numbers of this form, with these restrictions, are shown not to be the sum of two biquadrates either by using lemmas, small moduli, or in stubborn cases by examining all representations as sums of two squares. The extensive calculations involved do not invite further efforts in this direction. *D. H. Lehmer.*

Carlitz, L. q-Bernoulli numbers and polynomials. Duke Math. J. **15**, 987–1000 (1948).

The author defines a set of numbers η_m by means of the symbolic formula $(q\eta+1)^m=\eta_m$, $m>1$, $\eta_0=1$, $\eta_1=0$, in which, after expansion, the exponents of η are degraded to subscripts; and a set of polynomials $\eta_m(x)=\eta_m(x, q)$ in q^x such that $\eta_m(x+1)-\eta_m(x)=mq^x[x]^{m-1}$, $\eta_m(0)=\eta_m$, where $[x]=(q^x-1)/(q-1)$. Next he defines a set of numbers β_m by means of $\beta_m=\eta_m+(q-1)\eta_{m+1}$ and a set of polynomials $\beta_m(x)=\beta_m(x, q)$ such that $q^x\beta_m(x)=\eta_m(x)+(q-1)\eta_{m+1}(x)$, $\beta_m(0)=\beta_m$. For $q=1$, β_m reduces to the Bernoulli number B_m; η_m, however, does not remain finite. A detailed study of the η's and β's is made. In particular, explicit expressions are derived for β_m in terms of certain generalized Stirling numbers of the second kind. The main result of the paper is the following partial generalization of the Staudt-Clausen theorem: $\beta_m=\sum_{k=2}^{m+1} N_{m,k}(q)/F_k(q)$, where $F_k(q)$ denotes the cyclotomic polynomial and $N_{m,k}(q)$ is a polynomial in q which satisfies

$$(q-1)^{m-1}N_{m,k}(q)\equiv qF_k'(q)\sum_{1\leqq sk\leqq m+1}(-1)^{m+1+sk}\binom{m}{sk-1} \pmod{F_k(q)}.$$

The paper concludes with an analogous development of a q-generalization of the Euler numbers. *A. L. Whiteman.*

Carlitz, L. Finite sums and interpolation formulas over $GF[p^n, x]$. Duke Math. J. **15**, 1001–1012 (1948).

Let A and M denote polynomials in an indeterminate x with coefficients in the Galois field $GF(p^n)$. It is first proved that a polynomial $f(t)$, in another indeterminate t, satisfies $f(t+A)=f(t)$ for all A of degree less than m if and only if $f(t)=g(\psi_m(t))$, where $g(t)$ is also a polynomial and $\psi_m(t)=\Pi(t-M)=\sum_{i=0}^{m}(-1)^{m-i}[m/i]t^{p^{ni}}$, $\deg M<m$, $\psi_0(t)=t$, and where $[m/i]=F_m/F_iL_{m-i}^{p^{ni}}$, $[m/0]=F_m/L_m$, $[m/m]=1$, and $F_m=[m][m-1]^{p^n}\cdots[1]^{p^{n(m-1)}}$, $F_0=1$, $L_m=[m][m-1]\cdots[1]$, $L_0=1$, $[m]=x^{p^{nm}}-x$. Next a discussion is given of the equation $\sum h(t+A)=(-1)^mF_mg(\psi_m(t))/L_m$, $\deg A<M$. Sums of the type $\sum h(M)M(t)$, $\deg M\leqq m$, are also studied and some criteria for the vanishing of $\sum M^r$, M primary, $\deg M=m$, and $\sum M^r$, M primary, $\deg M<m$ $(r>0)$, are derived. In the remainder of the paper various interpolation formulas are established of which the following is typical: $(-1)^mF_mg(t)/L_m=\sum\psi_m(t)g(M)/(t-M)$, $\deg M<m$, $\deg g(t)<p^{nm}$. This result implies the theorem that a polynomial $g(t)$ of degree less than p^{nm} is integral-valued if and only if $g(M)$ is integral for all M of degree less than m. *A. L. Whiteman* (Los Angeles, Calif.).

Gorškov, D. S. On the Euclidean algorithm in real quadratic fields. Učenye Zapiski Kazan. Univ. **101**, kn. 3, 31–37 (1941). (Russian)

Gorškov, D. S. Real quadratic fields without a Euclidean algorithm. Učenye Zapiski Kazan. Univ. **101**, kn. 3, 37–42 (1941). (Russian)

These papers contain several results which were interesting at the time of publication. They are, however, all

uperseded by the results of Davenport [cf. the review of a paper by Inkeri, these Rev. **10**, 15], since the problem concerning the Euclidean algorithm in a quadratic field has been completely solved. *L. K. Hua* (Urbana, Ill.).

Ore, Oystein. On the averages of the divisors of a number. Amer. Math. Monthly **55**, 615–619 (1948).

Let $d(n)$ denote the number of divisors of n and $\sigma(n)$ the sum of the divisors of n. The author defines the arithmetic mean of the divisors, $A(n)=\sigma(n)/d(n)$, the harmonic mean by $1/H(n)=(\sum 1/k)/d(n)$ which simplifies to $H(n)=nd(n)/\sigma(n)$, and the geometric mean by $G(n)=(\prod_{k|n}k)^{1/d(n)}$. From this it follows that $H(n)A(n)=n=G(n)^2$. The author now investigates the question of determining the integers for which $A(n)$, $G(n)$, and $H(n)$ are integers. The question seems difficult, and only partial results are given. The famous conjecture concerning perfect numbers is generalized to "a number with integral harmonic mean of divisors must be even."

R. Bellman (Stanford University, Calif.).

Barbour, J. M. Music and ternary continued fractions. Amer. Math. Monthly **55**, 545–555 (1948).

This paper discusses the problem of the division of the octave into equally tempered intervals by means of continued fractions. There is an interesting historical account of the early and modern attempts to achieve a perfect tuning system. Objections to the 12 tone equally tempered system in common use have been raised by many writers. The chief trouble with the system is the sharpness of the major third. In fact with equal temperament the ratio $(\log \frac{5}{4})/\log 2=.3219281$ is to be compared with $\frac{4}{12}=.333\cdots$. The latter is too high by more than 3 percent. For the fifth the ratio $(\log \frac{3}{2})/\log 2=.5849625$ is to be compared with $\frac{7}{12}=.58333\cdots$. A division of the octave into more than 12 intervals is necessary to improve these approximations. Previous attempts have employed ordinary continued fractions rational approximations to $(\log \frac{3}{2})/\log 2$, obtained as usual from convergents and semi-convergents. The approximate third is then made to fit as well as possible.

The author submits that this unsymmetrical procedure is opposed to the fact that today's music is based on the major triad and holds that the irrational ratios (1) $\log \frac{5}{4}:\log \frac{3}{2}:\log 2$ should be approximated simultaneously without favoring one ratio over another. This calls for the use of ternary continued fractions. However, the ordinary Jacobi algorithm converges too rapidly. This means that one very soon reaches approximations involving the division of the octave into more than 100 parts. The author gives two modifications of the Jacobi algorithm which reduce its rate of convergence and thus obtains as many as 16 best approximations $A:B:C$ to (1) with $C<100$. The tenth of these is the standard 4:7:12. Others are 6:11:19, 7:13:22, 10:18:31. It is suggested that these modifications of Jacobi's algorithm have other applications. *D. H. Lehmer.*

Heilbronn, H. On the distribution of the sequence $n^2\theta$ (mod 1). Quart. J. Math., Oxford Ser. **19**, 249–256 (1948).

The general problem mentioned in the title is not treated, but rather the special question, how closely can the fractional part of $n^2\theta$ approach zero? This question was first posed by Hardy and Littlewood [Acta Math. **37**, 155–191 (1914)]; they conjectured that for all real θ it is possible to make $|n^2\theta-m|<c/N$ for every positive integer N, for suitable integers n, m with $1\leqq n\leqq N$. Here c is an absolute constant. Vinogradov [Bull. Acad. Sci. URSS [Izvestiya Akad. Nauk SSSR] (6) **21**, 567–578 (1927)] proved a general theorem, of which a special case is that this statement holds if c/N is replaced by $c(\eta)/N^{\frac{2}{5}-\eta}$, for every positive η. ($c(\eta)$ depends only on η.) The present work improves this to $c(\eta)/N^{\frac{1}{2}-\eta}$ by an adaptation of Vinogradov's method.

W. J. LeVeque (Cambridge, Mass.).

Mordell, L. J. The minimum of a definite ternary quadratic form. J. London Math. Soc. **23**, 175–178 (1948).

More than a century ago, Gauss proved that, of all positive definite ternary quadratic forms of minimum value 1 (for integral values of the variables, not all zero), those of minimum determinant are equivalent to $x^2+xy+y^2+yz+z^2$. Geometrically, this means that, of all sphere-packings (in ordinary space) where the centers of the spheres form a lattice, the densest is the cubic close-packing. The treatments by Gauss, Dirichlet, Hermite, Korkine and Zolotareff, and Minkowski are all quite complicated. The present paper contains a proof using nothing but elementary algebra.

H. S. M. Coxeter (New York, N. Y.).

Rankin, R. A. On sums of powers of linear forms. III. Nederl. Akad. Wetensch., Proc. **51**, 846–853 = Indagationes Math. **10**, 274–281 (1948).

Let $1\leqq\beta\leqq 2$, $n\geqq 2$; let $L_j=\sum_{k=1}^{n}a_{jk}x_k$ $(j=1,\cdots,n)$ be n real linear forms with determinant $D\neq 0$. Let $g_\beta(x_1,\cdots,x_n)=(\sum_{j=1}^{n}|L_j|^\beta)^{1/\beta}$ and let $M(g_\beta)$ be the lower bound of g_β for all systems of integers $x_1,\cdots,x_n$ other than $0,\cdots,0$; then (1) $M(g_\beta)\leqq n^\alpha|D|^{1/n}/A_\delta'(\alpha,\delta)$, where

$$A_\delta'(\alpha,\delta)=2^{-\alpha}n^\delta\left(\frac{2-\delta}{1-\delta}\right)^{\alpha-\delta}\left(\frac{(1+n\delta)(\alpha+\gamma-2)}{(1-\delta)I_\gamma|D|}\right)^{-1/n};$$

here $\alpha=\beta^{-1}$, $\gamma=\delta^{-1}$ and δ is any number such that $\frac{1}{2}\leqq\delta\leqq\alpha\leqq 1$, $\delta\leqq\frac{1}{3}(1+\alpha)$; further $I_\gamma=2^n\Gamma^n(1+\delta)|D|^{-1}\Gamma^{-1}(1+n\delta)$. For large values of n and for $\delta=\frac{1}{2}$, (1) has asymptotically the same degree of precision as an analogous formula given by E. Hlawka [Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **156**, 247–254 (1948); these Rev. **10**, 236]. But if $\alpha>0.77673\cdots$, there are values of δ which are more advantageous than $\delta=\frac{1}{2}$. The improvement is about 0.7% for $\alpha=0.9$ and about 2% for $\alpha=1$. See also Loo-Keng Hua [Bull. Amer. Math. Soc. **51**, 537–539 (1945); these Rev. **7**, 51], P. Mullender [Amsterdam thesis, 1945; these Rev. **9**, 335] and paper II of the present series [forthcoming]. [In the last formula on p. 853 replace $1/n$ by $-1/n$.]

V. Jarník (Prague).

Hlawka, Edmund. Inhomogene Linearformen in algebraischen Zahlkörpern. Akad. Wiss. Wien, S.-B. IIa. **155**, 63–73 (1947).

The author extends the method of Siegel [see H. Davenport, Acta Arith. **2**, 262–265 (1937)], relating to the product of n nonhomogeneous linear forms, to the case when the homogeneous parts of the linear forms are those arising from an "Ordnung" in an algebraic number-field, and its conjugates. The formulation of the main theorem is not altogether clear to the reviewer, and is incomplete as regards the reference to the determinant of the forms.

H. Davenport (London).

Chalk, J. H. H., and Rogers, C. A. The critical determinant of a convex cylinder. J. London Math. Soc. **23**, 178–187 (1948).

Let D be a plane convex domain, symmetric with respect to the origin O, and K the convex cylinder of all points

(x, y, z) for which $|z| \leqq 1$ and (x, y) is a point of D. Let $\Delta(D)$ and $\Delta(K)$ denote the lower bounds of the determinants of all the D-admissible lattices and K-admissible lattices, respectively. The object of this paper is to show $\Delta(D) = \Delta(K)$. The authors prove that a critical lattice of K may always be deformed continuously into a critical lattice with three linearly independent boundary points for which $|z| = 1$. These points are shown to generate the lattice and a study of their projections on the plane $z = 0$ leads to the result. This result is then used to show that the density of the closest lattice packing of 3-space by bodies congruent to K is the same as the corresponding density of the closest lattice packing of the plane by domains congruent to D. The same problem is the subject of the paper reviewed below.
D. Derry (Vancouver, B. C.).

Yeh, Yenchien. Lattice points in a cylinder over a convex domain. J. London Math. Soc. **23**, 188–195 (1948).

The author discusses the same problem as that of the paper reviewed above. By use of a result of Minkowski the author shows it is sufficient to consider those K-admissible lattices generated by the three vectors OP_1, OP_2, OP_3 with P_1, P_2, P_3 on K and for which either the three z-coordinates are all 1 or all nonnegative and less than 1. The projections of these three vectors on the plane $z = 0$ are shown to define certain D-admissible lattices by means of which the lower bound for the volume of the fundamental parallelepiped generated by P_1, P_2, P_3 is obtained. *D. Derry.*

Varnavides, P. On lattice points in a hyperbolic cylinder. J. London Math. Soc. **23**, 195–199 (1948).

Let K be the cylinder of all points (x, y, z) for which $|xy| \leqq 1$, $|z| \leqq 1$. It is shown that the lower bound $\Delta(K)$ of the determinants of admissible lattices of K is $\sqrt{5}$, which is the same as the corresponding number for the plane domain $|xy| \leqq 1$. *D. Derry* (Vancouver, B. C.).

Mullender, P. Lattice points in non-convex regions. I. Nederl. Akad. Wetensch., Proc. **51**, 874–884 = Indagationes Math. **10**, 302–312 (1948).

The author gives [as an application of a well-known theorem of Blichfeldt] the following generalization of a theorem of L. J. Mordell [same Proc. **49**, 773–781, 782–792 = Indagationes Math. **8**, 476–484, 485–495 (1946); these Rev. **8**, 369] concerning nonconvex regions. Let S_1 and S_2 denote the p- and q-dimensional spaces of points $x = (x_1, \cdots, x_p)$ and $y = (y_1, \cdots, y_q)$, respectively, and let S denote the $n = (p+q)$-dimensional space of points $(x, y) = (x_1, \cdots, x_p, y_1, \cdots, y_q)$. Let M and N be convex regions which are closed, bounded and symmetric about the origin O in the spaces S_1 and S_2, respectively. Let P and Q denote the p- and q-dimensional volumes of the regions M and N, respectively. Let $f(\mu)$ be a positive, steadily decreasing function of μ, defined for $0 < \mu < a$, with steadily increasing derivative. Put $f(0) = \lim_{\mu \to 0+} f(\mu)$ and let $f(a) = \lim_{\mu \to a-} f(\mu) = 0$, where $f(0)$ and a may be infinite. Let ξ be any number for which $0 < \xi < a$, $f(\xi) + \xi f'(\xi) > 0$. Let $g(t)$ be a differentiable function of t, defined for $0 \leqq t \leqq \frac{1}{2}\xi$, with $g(\frac{1}{2}\xi) = \frac{1}{2}\xi$, $g'(t) < 0$. Let $\beta = g(0) < a - \frac{1}{2}\xi$. Let $t = h(\bar{t})$ be the inverse function of $\bar{t} = g(t)$. Put $\varphi(\mu) = \frac{1}{2}f(\xi) - \int_\mu^{\frac{1}{2}\xi} f'(t + g(t))dt$ for $0 \leqq \mu \leqq \frac{1}{2}\xi$; $\varphi(\mu) = \frac{1}{2}f(\xi) + \int_{\frac{1}{2}\xi}^{\mu} f'(h(t) + t)dt$ for $\frac{1}{2}\xi \leqq \mu \leqq \beta$; $\varphi(\mu) = f(\mu) - \varphi(0)$ for $\beta \leqq \mu \leqq a$. Let R be the set of the points $(x, y) \varepsilon S$ for which there exist two numbers μ, ν such that $0 \leqq \mu \leqq a$, $0 \leqq \nu \leqq f(\mu)$, $x \varepsilon \mu M$, $y \varepsilon \nu N$. Let $0 \neq |\Delta| \leqq pPQ\int_0^\alpha (\varphi(t))^q t^{p-1} dt$, where α is defined by $\varphi(\alpha) = 0$. Then R contains a point other than O of any lattice with determinant Δ. In the second part of this paper the author gives some applications of this theorem. *V. Knichal* (Prague).

Chowla, S. Improvement of a theorem of Linnik and Walfisz. Proc. London Math. Soc. (2) **50**, 423–429 (1949).

If χ is a real primitive character modulo k and $L(s, \chi) = \sum_{n=1}^\infty \chi(n)n^{-s}$, then it is known [cf. Siegel, Acta Arith. **1**, 83–86 (1935)] that for any positive ϵ we have $Ak^{-\epsilon} < L(1, \chi) < B \log k$, where A is a positive number depending only on ϵ and B is an absolute constant. Linnik [C. R. (Doklady) Acad. Sci. URSS (N.S.) **37**, 122–124 (1942); these Rev. **5**, 142] and Walfisz [Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **11**, 57–71 (1942); these Rev. **5**, 254] have both proved that there is a positive constant C such that for infinitely many k we have $L(1, \chi) < C(\log \log k)^{-\frac{1}{2}}$ if χ is a real primitive character modulo k. In the present paper the author proves that there are infinitely many k such that $L(1, \chi) < \{1 + o(1)\}\pi^2/(6e^\gamma \log \log k)$ for χ a real primitive character modulo k, γ being Euler's constant. The proof is rather simple; it is a sharpening of the method used earlier by the author [Proc. Benares Math. Soc. **5**, 23–27 (1943) = Proc. Lahore Philos. Soc. **6**, no. 1, 9–12 (1944); these Rev. **7**, 243] to prove a result weaker than the Linnik-Walfisz theorem. The sharpening consists of choosing the number g of the earlier paper as $[\log x/(\log \log x)^3]$. [Correction: the inequality signs should be reversed in (21), in (iii) of theorem 1, and in theorem 2.] *P. T. Bateman.*

Chowla, S. On the class-number of the corpus $P(\sqrt{-k})$. Proc. Nat. Inst. Sci. India **13**, 197–200 (1947).

This paper contains a result similar to that of the paper reviewed above, but in the opposite direction. The author has proved earlier [Math. Z. **38**, 483–487 (1934)] that there is a positive constant D such that for infinitely many k the inequality $L(1, \chi) > D \log \log k$ holds if χ is a real primitive character modulo k. Walfisz [Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **11**, 57–71 (1942); these Rev. **5**, 254] improved this result slightly by showing that for infinitely many k (*) $L(1, \chi) > \{1 + o(1)\}e^\gamma \log \log k$ if χ is a real primitive character modulo k, γ being Euler's constant. In the present paper the author gives a simpler proof of (*); the method is very much like that used in the paper above.

Corrections. The theorem that $\sum_{n=u}^{v} \chi(n) = O(k^{\frac{1}{2}} \log k)$ for any non-principal character χ modulo k is erroneously attributed to Davenport. Actually this theorem seems to have been first proved generally by Landau [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. **1918**, 79–97, pp. 85–86]; the basic case of a primitive χ was first given by Pólya [ibid. 21–29]. Also χ is consistently written as x (which is also used otherwise) and "m odd" is consistently written as "mod d." *P. T. Bateman* (Princeton, N. J.).

Romanov, N. P. Concerning the distribution of prime numbers. Mat. Sbornik N.S. **23**(65), 259–278 (1948). (Russian)

Let $\Lambda(n)$, $\mu(n)$ and $\varphi(n)$ be the functions of von Mangoldt, Möbius and Euler, respectively, and let

$$Q_n(x) = -\mu(n)x + \sum_{d|n} \mu(n/d) \frac{dx^d}{1 - x^d} = \sum_\rho \frac{\rho^2 x^2}{1 - \rho x},$$

where ρ runs through the $\varphi(n)$ primitive nth roots of unity. In the course of the paper the following result of Hardy and

Littlewood is applied several times:

$$\text{(A)}\qquad \lim_{r\to 1-0}(1-r)\sum_{n=1}^{\infty}c_n r^n=A,\qquad c_n\geqq 0,$$

implies $\lim_{N\to\infty} N^{-1}\sum_{n=1}^{N}c_n=A$. The author proves, by elementary methods, the formula

$$\text{(B)}\qquad \sum_{m=1}^{\infty}m^{-1}\varphi(m)\Lambda(m)x^m=\sum_{n=1}^{\infty}\{\mu(n)/\varphi(n)\}Q_n(x),\qquad |x|<1.$$

This is achieved by considering partial sums of the second series which are shown to tend to the first series as the number of terms included is increased. The second series, when multiplied by $1-|x|$, is uniformly convergent for $x=re^{2\pi il/k}$, $(k,l)=1$, as $r\to 1-0$, and it is shown that the prime number theorem, and the corresponding result (C) $\pi(k,l;x)\sim x/\{\varphi(k)\log x\}$ for the number of primes in an arithmetic progression, can be deduced quite simply from this fact. However, the uniform convergence cannot be deduced from the elementary proof of (B) which is given, and the author has to make use of complex variable methods which are of the same depth as the prime number theorem, as proved by Hadamard and de la Vallée Poussin, in order to prove that the convergence is uniform. The analysis is complicated and an adequate account of the methods used cannot be given here.

Other applications of the same ideas are given, and (B) is generalised to sequences of exponents $n_1, n_2, \cdots$ in place of $1, 2, \cdots$. An alternative method of deducing (C) is sketched. This is based on the result

$$\text{(D)}\qquad \limsup_{x\to\exp(2\pi il/k)}(1-|x|)|F_s(x)-F(x)|\leqq A(s-1),$$

where $F(x)=\sum_{n=1}^{\infty}\Lambda(n)x^n$ and

$$F_s(x)=\sum_{n=1}^{\infty}\lambda(n,s)x^n=\zeta(s)\sum_{n=1}^{\infty}x^n\prod_{p|n}(1-p^{1-s}).$$

Here $\zeta(s)$ denotes the Riemann zeta-function and $\lambda(n,s)\to\Lambda(n)$ as $s\to 1$. It is shown that it is sufficient to prove that

$$\text{(E)}\qquad \sum_{n\leqq N}\frac{d}{ds}\lambda(n,s)=O(N)\qquad (n\equiv l(\operatorname{mod} k)),$$

uniformly for $1\leqq s\leqq 2$. Only the first part of the proof of (E) is given in full, so that it is not clear to the reviewer whether the complete proof uses complex variable methods or not. If it does not, the author has obtained an elementary proof of (C) and the full details would be of interest.

The application of the method to problems such as that of Goldbach for two primes is sketched. Here again everything hinges on the uniform convergence of the series obtained, and it is shown that, if this is assumed, the conjectured asymptotic formulae follow. The paper is a sequel to, but is largely independent of, a previous paper [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **10**, 3–34 (1946); these Rev. **8**, 9]. *R. A. Rankin.*

Turán, Paul. On some approximative Dirichlet-polynomials in the theory of the zeta-function of Riemann. Danske Vid. Selsk. Mat.-Fys. Medd. **24**, no. 17, 36 pp. (1948).

(I) If the partial sums $U_n(s)=\sum_1^n\nu^{-s}$ of the series for $\zeta(s)$ $(s=\sigma+it)$ have no zeros in $\sigma>1$ $(n>n_0)$, then $\zeta(s)$ has no zeros in $\sigma>\frac12$ (i.e., the Riemann hypothesis is true). Proof. By a theorem of Bohr (based on the linear independence of the numbers $\log p$) the hypothesis $U_n(s)\neq 0$ $(\sigma>1,\ n>n_0)$ is equivalent to $W_n(s)\equiv\sum_1^n\lambda(\nu)\nu^{-s}\neq 0$ $(\sigma>1,\ n>n_0)$, and so implies $W_n(\sigma)\geqq 0$ $(\sigma\geqq 1,\ n>n_0)$, where $\lambda(n)$ is Liouville's function; and this implies, by a theorem of Landau, that the function on the right of the identity

$$\int_1^{\infty}x^{-s}W_{[x]}(1)dx=\frac{\zeta(2s)}{(s-1)\zeta(s)},\qquad \sigma>1,$$

being regular along the stretch $s>\frac12$ of the real axis, is regular in the half-plane $\sigma>\frac12$. Among refinements and extensions the following may be mentioned. (IV) If, for $n>n_0$, $U_n(s)$ omits in $\sigma>1+Kn^{\vartheta-1}$ a real value c_n with $-K_1n^{\vartheta-1}\leqq c_n\leqq K_1n^{\vartheta-1}$, then $\zeta(s)\neq 0$ for $\sigma>\vartheta$ $(\frac12\leqq\vartheta<1)$. (X) If, for some character $\chi(n)$ mod k, $\sum_1^n\chi(\nu)\nu^{-s}\neq 0$ for $\sigma>1$, then $\zeta(s)\neq 0$ for $\sigma>\frac12$. No implication in the opposite direction is suggested. The hypothesis $U_n(s)\neq 0$ $(\sigma>1)$ is tested in various ways. (1) A possible threat from Knopp's theorem, that every point of $\sigma=1$ is a cluster point of zeros of $\{U_n(s)\}$, is countered by a proof (VI) that the clustering is wholly from the left, at any rate if $|t|\geqq\tau_0$. (2) The inequality $W_n(1)>0$ has been verified up to $n=1000$ by a group of Danish mathematicians.

[Note by the reviewer. The plausibility of the hypothesis $U_n(s)\neq 0$ $(\sigma>1)$ (or the weaker hypothesis of (IV) with $\vartheta=\frac12$) is somewhat diminished by the observation that it implies, by way of $W_n(1)\geqq 0$ (or $W_n(1)>-K_2n^{-\frac12}$), not merely that the complex zeros of $\zeta(s)$ are of the form $\frac12\pm i\gamma_n$ ($\gamma_n>0$ and distinct), but also that the γ_n are linearly dependent. For the parallel discussion of Pólya's hypothesis $W_n(0)\leqq 0$ $(n\geqq 2)$ see Ingham, Amer. J. Math. **64**, 313–319 (1942); these Rev. **3**, 271.] *A. E. Ingham.*

ANALYSIS

Davenport, H., and Pólya, G. On the product of two power series. Canadian J. Math. **1**, 1–5 (1949).

A positive sequence $\{c_n\}$ is said to be logarithmically convex if $c_n^2\leqq c_{n-1}c_{n+1}$ for $n=1, 2, \cdots$. A theorem of Kaluza [Math. Z. **28**, 161–170 (1928)] asserts that if $\{a_n\}$ is logarithmically convex, and $(\sum_0^{\infty}a_nx^n)^{-1}=b_0-b_1x-b_2x^2-\cdots$, then $\{b_n\}$ is positive. This paper treats a similar type of question. Let positive sequences $\{\alpha_n\}$ and $\{\beta_n\}$ be defined by $\sum_0^{\infty}\alpha_nx^n=(1-x)^{-\alpha}$, $\sum_0^{\infty}\beta_nx^n=(1-x)^{-\beta}$, where $\alpha>0, \beta>0$, and $\alpha+\beta=1$. Let $\{a_n\}$ and $\{b_n\}$ be positive sequences, and define a sequence $\{w_n\}$ by $\sum_0^{\infty}w_nx^n=(\sum_0^{\infty}a_n\alpha_nx^n)(\sum_0^{\infty}b_n\beta_nx^n)$. Then, if $\{a_n\}$ and $\{b_n\}$ are both monotone increasing or decreasing, so is $\{w_n\}$, and if both $\{a_n\}$ and $\{b_n\}$ are logarithmically convex, so is $\{w_n\}$. If $\phi(t)$ is positive and integrable on $[0, h]$, then $\int_0^h\phi(t)(1-xt)^{-\alpha}dt=\sum_0^{\infty}a_n\alpha_nx^n$, where $a_n=\int_0^h\phi(t)t^ndt$, and $\{a_n\}$ is logarithmically convex. The authors apply these results, together with the theorem of Kaluza, to show that a certain function is monotone.

R. C. Buck (Providence, R. I.).

Macintyre, Sheila Scott. A functional inequality. J. London Math. Soc. **23**, 202–209 (1948).
Macintyre, A. J. Note on the preceding paper. J. London Math. Soc. **23**, 209–211 (1948).

These papers give conditions on a real function $f(x)$ and its derivatives to ensure that $f(x)\leqq\sin x$ in a certain interval. The problem was initiated by E. M. Wright [same J. **22** (1947), 205–210 (1948); these Rev. **9**, 415]. The first paper

uses the expansion of $f(x)$ in terms of the Gontcharoff polynomials to prove two new theorems of this kind; the second of these will illustrate the general character of the results. It is as follows: let a_n be max $|f^{(n)}(x)|$ for $0\leqq x\leqq \frac{1}{2}\pi$; let $0\leqq x\leqq \eta<\frac{1}{2}\pi$; let $(-1)^{\frac{1}{2}(n-1)}f^{(n)}(0)\leqq 1$ for odd n, $(-1)^{\frac{1}{2}n}f^{(n)}(\eta)\leqq \sin \eta$ for even n, $f(0)\leqq 0$; let $\liminf n^{-1}\log a_n\leqq 0$; then $f(x)\leqq \sin x$ in $(0, \eta)$.

The second paper uses one of the lemmas of the first to prove a direct generalization of one of Wright's theorems.

R. P. Boas, Jr. (Providence, R. I.).

Boas, R. P., Jr., and Chandrasekharan, K. Correction: Derivatives of infinite order. Bull. Amer. Math. Soc. 54, 1191 (1948).

The paper appeared in same vol., 523–526 (1948); these Rev. 10, 21.

Theory of Sets, Theory of Functions of Real Variables

***Šanin, N. A. On the product of topological spaces.** Trudy Mat. Inst. Steklov. 24, 112 pp. (1948). (Russian)

A calibre of a space is a cardinal number $m>1$, infinite in all cases of interest, such that every family of power m of open subsets of the space has a subfamily of like power whose intersection is not empty. Every cardinal which exceeds the power of the space itself is trivially a calibre. The study is principally concerned with the calibres of a direct product of a finite or infinite number of spaces of assigned calibres. A motivation of this study is the theorem of Szpilrajn [C. R. (Doklady) Acad. Sci. URSS (N.S.) 31, 525–527 (1941); these Rev. 3, 57] to the effect that every uncountable collection of open subsets of a product of spaces, each of which has a countable base, has at least one pair of intersecting sets. Concepts related to that of calibre are applied to a study of the diadic bicompacta: these are continuous images of a product of compacta.

L. Zippin (Flushing, N. Y.).

Sierpiński, W. Sur quelques propriétés du nombre $2^{\aleph_0}$. Mathematica, Timişoara 23, 60–64 (1948).

The author proves without using the axiom of choice various theorems from the arithmetic of cardinal numbers (e.g., if m is a nonfinite cardinal, then $m\leqq 2^{\aleph_0}$ implies $2^{\aleph_0}\leqq 2^m$). These theorems were announced, without proofs, by A. Tarski [Lindenbaum and Tarski, C. R. Soc. Sci. Lett. Varsovie. Cl. III. 19, 299–330 (1926)], but the following lemma, which is used in the proofs, is new. Any nonfinite set of real numbers can be constructively decomposed into an infinite sequence of nonempty disjoint subsets.

B. Jónsson (Providence, R. I.).

Sierpiński, Wacław. Sur les translations des ensembles linéaires. Fund. Math. 35, 159–164 (1948).

If E is a set of real numbers, its translation by a is the set of $x+a$ for all x in E. How many distinct translations does a given set have? (Distinct translations might overlap.) If $N(E)$ is the number of distinct translations, then $N(E)$ is infinite unless E is empty or the set of all real numbers, and for every cardinal m with $\aleph_0\leqq m\leqq 2^{\aleph_0}$ there are sets E for which $N(E)=m$. If $N(E)=\aleph_0$ then E is nonmeasurable, has the property (*): the outer measure of E in every interval is the length of the interval. Every E which is not a null-set and for which $N(E)<2^{\aleph_0}$ has this property (*).

I. Halperin (Kingston, Ont.).

Marczewski, Edward. Ensembles indépendants et leurs applications à la théorie de la mesure. Fund. Math. 35, 13–28 (1948).

The author gives details of proofs of results announced earlier [C. R. Acad. Sci. Paris 207, 768–770 (1938)] and carries on that work. The essential idea is to discuss families of sets with the property that the intersection of finitely many (or sometimes of enumerably infinitely many) of the sets is never empty, with particular emphasis on the measures that can be defined on the fields of sets generated by these families. More general results have been obtained (in more complicated ways) by the author and Banach [abstract by Marczewski, Ann. Soc. Polon. Math. 19 (1946), 247–248 (1947); forthcoming paper by Banach].

J. L. Doob (Urbana, Ill.).

Doob, J. L. On a problem of Marczewski. Colloquium Math. 1, 216–217 (1948).

If X is a measure space with measure m, such that $m(X)=1$, and if f and g are real-valued measurable functions on X, then f and g are called S-independent if (*) $m(f^{-1}(F)\cap g^{-1}(G))=m(f^{-1}(F))m(g^{-1}(G))$ whenever F and G are real Borel sets, and f and g are called K-independent if (*) holds whenever $f^{-1}(F)$ and $g^{-1}(G)$ are measurable. The author constructs a pair of S-independent functions which are not K-independent.

P. R. Halmos.

Jessen, B. On two notions of independent functions. Colloquium Math. 1, 214–215 (1948).

An example is given to show that two closely related definitions of independence of functions are not equivalent. A similar example was found by the reviewer [cf. the paper reviewed above].

J. L. Doob (Urbana, Ill.).

Hodges, J. L., Jr., and Horn, Alfred. On Maharam's conditions for measure. Trans. Amer. Math. Soc. 64, 594–595 (1948).

The authors prove that one of Maharam's conditions for the existence of a positive measure in a Boolean σ-algebra [Ann. of Math. (2) 48, 154–167 (1947); these Rev. 8, 321] is a consequence of the others.

P. R. Halmos.

Popruženko, J. Sur la représentation analytique de certaines classes des fonctions additives d'ensemble. Bull. Int. Acad. Polon. Sci. Cl. Sci. Math. Nat. Sér. A. Sci. Math. 1940–1946, 35–49 (1948).

The author's principal result may be formulated as follows. If μ is a regular Carathéodory outer measure on a topological space K having the power of the continuum, and if F is a real-valued, finitely additive set function defined on the class of all μ-measurable sets such that $F(\{p\})=0$ for each p in K and such that F is countably additive on the class of all Borel sets in K, then F has a unique representation in the form $F=\varphi+\psi$, where φ is absolutely continuous with respect to μ and where ψ is singular in the sense that every μ-measurable set may be written as a countable disjoint union of sets on which it vanishes identically. The proof uses the continuum hypothesis. As consequences the author deduces several propositions asserting essentially that certain conditions apparently weaker than absolute continuity are in fact equivalent to absolute continuity.

P. R. Halmos (Chicago, Ill.).

McMinn, Trevor J. Restricted measurability. Bull. Amer. Math. Soc. 54, 1105–1109 (1948).

If φ is an outer measure defined on the class A of all subsets of a set, and if F is a subclass of A, then a set a in

A is F-measurable if $\varphi(e\cap f)=\varphi(e\cap f\cap a)+\varphi(e\cap f\cap a')$, where $e\epsilon A$, $f\epsilon F$ and a' is the complement of a. A hereditary class F is called convenient if to each a in A with $\varphi(a)<\infty$ there corresponds an increasing sequence $\{f_n\}$ of F-measurable sets in F such that $\varphi(a-\bigcup_n f_n)=0$. The author's principal result is that if F is convenient, then the F-measurability of a set implies its A-measurability. An application of the result is a simple proof of the fact that if φ is an outer measure on a metric space such that $\varphi(a\cup b)=\varphi(a)+\varphi(b)$ whenever a and b are bounded sets at a positive distance from each other, then every open set is A-measurable.

P. R. Halmos (Chicago, Ill.).

Cesari, Lamberto. Funzioni continue a variazione limitata in un insieme. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) **2**, 127–145 (1946).

In studying continuous functions of bounded variation or absolutely continuous functions (always of one real variable), the author intends to combine the points of view of Jordan and of Banach and Vitali. Corresponding to these two points of view he uses the distinguishing marks V and W, respectively. Let $V(E,f)$ designate the total variation V of $f(x)$ in the set E (defined in the usual way). On the other hand, let $N(E,y)$ be the number of solutions of $f(x)=y$, $x\epsilon E$. If $N(E,y)$ is measurable in $[-\infty,\infty]$, then the author defines the total variation W of $f(x)$ in E by $W(E,f)=\int_{-\infty}^{\infty}N(E,y)dy$.

Let $f(x)$ always be continuous in the interval $I=[a,b]$. The author proves the following theorem. If $E\subset I$ is a Borel set, then $N(E,y)$ is measurable and $W(E,f)\leqq V(E,f)$. If, moreover, $W(E,f)<+\infty$ or $V(E,f)<+\infty$, then the author calls $f(x)$ of bounded variation W or V, respectively, in E. In the case that E is an open set, $W(E,f)$ reduces to the notion of total variation introduced by L. Tonelli. Correspondingly, the author calls $f(x)$ absolutely continuous W or V in $E(\subset I)$ if to every $\epsilon>0$ there corresponds a $\delta>0$ such that for every finite set of nonoverlapping intervals $I_s=[\alpha_s,\beta_s]$, $I_s\subset I$, $s=1,2,\cdots,n$, with $\sum|I_s|<\delta$, one has $\sum_{s=1}^{n}W(EI_s,f)<\epsilon$ or $\sum_{s=1}^{n}V(EI_s,f)<\epsilon$, respectively. All these notions are further investigated and, in particular, the author states the following theorem. If $f(x)$ is of bounded variation V in the Borel set $E(\subset I)$, then at almost all points x_0 of $\bar{E}$ the finite limit $D(x_0)=\lim\{f(x)-f(x_0)\}/(x-x_0)$, as $x\to x_0$, $x\epsilon\bar{E}$, exists. The proof is given here only for the case of a closed set E, while the general theorem is proved in the paper reviewed below. Moreover, $D(x)$ is summable in E and $W(E,f)\geqq\int_E|D(x)|dx$. In order that the equality sign holds here, it is necessary and sufficient that $f(x)$ is absolutely continuous W in E.

A. Rosenthal.

Cesari, Lamberto. Invertibilità in piccolo delle funzioni continue e teorema di derivazione. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) **3**, 99–112 (1947).

Using the notations of the paper reviewed above the author proves the following theorem. Let $y=f(x)$ be continuous in the interval $I=[a,b]$ and of bounded variation W in a Borel set $E\subset I$; let $c=\min f(x)$ and $d=\max f(x)$ in I. To every $\epsilon>0$ there corresponds a $\lambda>0$ such that, for every subdivision of I into a finite number of intervals $\delta_1,\cdots,\delta_n$ of lengths less than λ, (1) there exists in $[c,d]$ a system of open intervals Δ of measure less than ϵ and n sets $T_1,\cdots,T_n$ each of which consists of a finite number of closed disjoint intervals, such that $N(E\delta_i,y)$ is 1 if $y\epsilon T_i-T_i\Delta$ and 0 if $y\epsilon[c,d]-(T_i+\Delta)$ $(i=1,\cdots,n)$; (2) there exist n functions $x=\xi_i(y)$, $y\epsilon T_i-T_i\Delta$, continuous in the perfect set $T_i-T_i\Delta$, such that for every $y\epsilon T_i-T_i\Delta$ one has $y=f[\xi_i(y)]$ $(i=1,\cdots,n)$; (3) if one denotes by D the open set of points x in $[a,b]$ for which $f(x)\epsilon\Delta$, then for every $x\epsilon E\delta_i-E\delta_iD$ one has $x=\xi_i[f(x)]$ $(i=1,\cdots,n)$; (4) if $E\delta_i=\delta_i$ for a given i, then T_i is a single interval and for every $y\epsilon[c,d]-T_i$ one has $N(\delta_i,y)=0$. By means of this theorem the author then proves the general theorem on the derivative $D(x)$, already stated in the preceding paper, but proved there only for closed sets E.

A. Rosenthal.

Theory of Functions of Complex Variables

***Knopp, Konrad. Problem Book in the Theory of Functions. Volume 1. Problems in the Elementary Theory of Functions. Translated by Lipman Bers.** Dover Publications, Inc., New York, N. Y., 1948. viii+126 pp. $1.85.

Translated from the second German edition [de Gruyter, Berlin, 1931].

Fedorov, V. S. On the derivative of a complex function. Doklady Akad. Nauk SSSR (N.S.) **63**, 357–358 (1948). (Russian)

Let $\phi(t)=u(t)+iv(t)$ be a complex valued function of a real variable, continuous for $0\leqq t\leqq T$ and differentiable for $0<t<T$, let $\tau=[\phi(T)-\phi(0)]/T$, and let M denote the set of values in the complex plane taken on by the function $\phi'(t)$, $0<t<T$. It is shown that every line through the point τ in the complex plane has at least one point in common with M. The author makes several applications to complex functions of a complex variable. Thus it is shown that if the analytic function $w=f(z)$ is regular in a convex domain K, then a sufficient condition that $f(z)$ be schlicht is that there exist a real number α such that $e^{i\alpha}f'(z)$ is not real for any z in K; in particular, if $f'(z)$ is not real for any z in K then $f(z)$ must be schlicht.

E. F. Beckenbach.

Pompeiu, D. Du point à l'infini comme point singulier isolé. Bull. Math. Phys. Éc. Polytech. Bucarest **10** (1938–39), 13–19 (1940).

Montel, Paul. Sur une généralisation d'une théorème de Jacobi. Proc. Math. Phys. Soc. Egypt **3** (1947), 47–52 (1948).

Jacobi's theorem that an analytic function with three independent periods reduces to a constant is generalized in the following way. Let $\Delta_\lambda f$ denote the difference $f(z+\lambda)-f(z)$ and let all the triples h_α, k_β, l_γ $(\alpha,\beta,\gamma=1,\cdots,s)$ be linearly independent, i.e., there exists no relation $mh_\alpha+nk_\beta+pl_\gamma=0$ with integral m, n, p. Then the only solutions of the system

$$\text{(a)}\qquad \begin{aligned}\Delta_{h_1}\cdots\Delta_{h_s}f&=0,\\ \Delta_{k_1}\cdots\Delta_{k_s}f&=0,\\ \Delta_{l_1}\cdots\Delta_{l_s}f&=0\end{aligned}$$

are the polynomials in z of degree $s-1$. An analogous result holds for functions of a real variable; in this case, (a) is replaced by a system of two equations. A geometrical application is given.

Z. Nehari (St. Louis, Mo.).

Belardinelli, G. Su una serie di funzioni razionali. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **10(79)**, 97–102 (1946).

Let $\alpha_n=r_ne^{i\varphi_n}$, where $n+\frac{1}{2}<r_{n+1}<n+1$ and

$$|\varphi_n|\leqq|\varphi|<\pi/2.$$

Let $P_n(z)=\prod_1^n(z-\alpha_k)$: The author proves that, if $\sum_1^\infty C_nP_n(z)$ converges for $z=z_0$, it converges absolutely for all z such that, for some $\delta>0$ and all large n, $\Re[(z_0-z)e^{-i\varphi_n}]<-\delta-1$.
R. C. Buck (Providence, R. I.).

Belardinelli, G. Una applicazione della convergenza in media. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 10(79), 142–146 (1946).

Der Verfasser verwendet die Integraldarstellung der linearen analytischen Funktionale im Sinne von Fantappiè, $F[y(z)]=(2\pi i)^{-1}\int_C v(z)y(z)dz$, zu einer formalen Erweiterung der Theorie auf eine Klasse von komplexen Funktionen, deren Real- und Imaginärteil im Mittel gegen auf C einfach und quadratisch integrierbare Funktionen konvergieren, wenn man sich C auf einer geeigneten Kurvenschar nähert.
H. G. Haefeli (Boston, Mass.).

Sunyer Balaguer, F. On a class of transformations of the algorithms for summation of analytic series. Collectanea Math. 1, 109–143 (1948). (Spanish)

Generalizing a previous paper [C. R. Acad. Sci. Paris 208, 409–411 (1939)] the author constructs a class of methods of analytic extension, and discusses additional ones formed by composition of two or more methods, and by merging two or more limiting processes. *R. C. Buck.*

Broman, Arne. Conformal mapping and convergence of a power series. Proc. Nat. Acad. Sci. U. S. A. 34, 605–610 (1948).

The author discusses the following problem: Is it possible for a power series $w=f(z)=\sum_{n=0}^\infty a_nz^n$ to converge at every point of $|z|=1$, if $w=f(z)$ maps $|z|<1$ onto the universal covering surface of a finite multiply-connected domain? A simple argument shows that the answer is negative if, in addition to its outer boundary, D is bounded by more than one point. The author then discusses the exceptional case, in which D consists of one closed continuum C and one isolated point w_0. If, in particular, C is the circle $|w|=1$ and w_0 its center, the mapping function is of the form $f(z)=\exp[(z+1)(z-1)^{-1}]$. It is shown that in this case the power series expansion of $f(z)$ converges at every point of $|z|=1$. While this follows, for $z\neq1$, from classical theorems, the point $z=1$ requires special handling. *Z. Nehari.*

Milloux, H. Une application de la théorie des familles normales. Bull. Sci. Math. (2) 72, 12–16 (1948).

By means of normal families, the author establishes the existence of a function $A(\epsilon)$ of the positive variable ϵ, such that $A(\epsilon)\to0$ as $\epsilon\to0$, and with the following property. If the analytic function $f(z)$ is regular and satisfies $|f(z)|<1$ for $|z|<1$, and there is a curve C from $z=0$ to the circumference $|z|=\frac12$ such that on C we have $|f(z)|<\epsilon$, then throughout $|z|<\frac12$ we must have $|f(z)|<A(\epsilon)$. By a theorem of Stieltjes, if a sequence of bounded analytic functions $f_n(z)$, regular in a domain D, converges to zero in a subdomain D' of D, then the sequence converges uniformly to zero in the interior of D. The same conclusion holds if the sequence converges to zero on a curve C' in D. As an application of the above result, it is pointed out that D' or C' need not be fixed but might vary, under suitable restrictions, with n. But it is shown by an example that the similar theorem of Vitali, with D' or C' replaced by a set E' of points having a limit point interior to D, cannot be extended in a certain analogous way.
E. F. Beckenbach (Los Angeles, Calif.).

Shah, S. M. The maximum term of an entire series. III. Quart. J. Math., Oxford Ser. 19, 220–223 (1948).

[For papers I and II cf. Math. Student 10, 80–82 (1942); J. Indian Math. Soc. (N.S.) 9, 54–55 (1945); these Rev. 4, 137; 8, 23.] Let $f(z)$ be an entire function of finite positive order ρ and lower order λ; let $\mu(r)$ denote the maximum term in the power series for $f(z)$, $\nu(r)$ its rank; let T and t denote the upper and lower limits of $r^{-\rho}\log M(r)$, γ and δ those of $r^{-\rho}\nu(r)$, c and d those of $\{\log\mu(r)\}/\nu(r)$. The author proves $\delta\leqq\gamma e^{-1+\delta/\gamma}\leqq\rho T\leqq\gamma$, $\delta\leqq\rho t\leqq\delta(1+\log\gamma/\delta)\leqq\gamma$, whence (*) $\rho T\leqq\gamma\leqq e\rho T$, $\delta\leqq\rho t$. The author constructs examples to show that the inequalities (*) and the preceding upper bound for ρt are best possible. *R. P. Boas, Jr.*

Leont'ev, A. F. On interpolation in the class of entire functions of finite order. Doklady Akad. Nauk SSSR (N.S.) 61, 785–787 (1948). (Russian)

Let $\{\lambda_n\}$ be a sequence of complex numbers of non-decreasing absolute value and let $\{a_n\}$ be a sequence such that $\limsup\log\log|a_n|/\log|\lambda_n|\leqq\rho$, $0<\rho<\infty$. The author gives the following set of necessary and sufficient conditions for the existence of an entire function $\omega(z)$, of order not exceeding ρ, such that $\omega(\lambda_n)=a_n$. (A) The exponent of convergence of $\{\lambda_n\}$ does not exceed ρ. (B) Let $F(z)$ be the canonical product with zeros at λ_n. Then

$$\limsup\{\log|\lambda_n|\}^{-1}\log\log|1/F'(\lambda_n)|\leqq\rho.$$

R. P. Boas, Jr. (Providence, R. I.).

Korevaar, J. An inequality for entire functions of exponential type. Nieuw Arch. Wiskunde (2) 23, 55–62 (1949).

For $f(x)$ of exponential type α, the author proves

$$|f(x+iy)|^p\leqq A_p\int_{-\infty}^{\infty}|f(x)|^pdx\cdot y^{-1}\sinh p\alpha y,$$

$1\leqq p<\infty$, with $A_p\leqq\pi^{-1}$. For $p=2$ the best value of A_p is $\frac12\pi^{-1}$, but the best value is not known for other p.
R. P. Boas, Jr. (Providence, R. I.).

Ahiezer, N. I. On the theory of entire functions of finite degree. Doklady Akad. Nauk SSSR (N.S.) 63, 475–478 (1948). (Russian)

Let $f(z)$ be an entire function of finite degree σ (i.e., of exponential type σ), such that $f(x)\geqq0$ for all real x. The author proves that $f(x)=|\Phi(x)|^2$, $-\infty<x<\infty$, where $\Phi(z)$ is an entire function of degree $\sigma/2$, all of whose zeros are in the upper half plane, if and only if

$$\sup_{R>1}\int_1^R x^{-2}\log|f(x)f(-x)|dx<\infty.$$

R. P. Boas, Jr. (Providence, R. I.).

Tchakaloff, L. Sur une représentation des fonctions entières d'ordre un et du type zéro. C. R. Acad. Bulgare Sci. Math. Nat. 1, no. 1, 13–16 (1948).

The results of this note also appeared in Doklady Akad. Nauk SSSR (N.S.) 58, 535–538 (1947); these Rev. 9, 422.
R. P. Boas, Jr. (Providence, R. I.).

Meĭman, N. N. On a class of entire functions. Doklady Akad. Nauk SSSR (N.S.) 62, 293–296 (1948). (Russian)

In the formula $F(z)=g(z)+ih(z)$, g and h are integral functions assuming real values on the real axis and without common zeros (the last assumption avoids nonessential complications). Say that $F(z)\epsilon B$, if $J(z)=(g(z)+ih(z))/(g(z)-ih(z))$

is bounded in $y>0$, $|z|>r_0$. Theorem 1: $F(z)\epsilon B$, if and only if the following conditions are satisfied:

$$g(z)=R(z)e^{u(z)}\prod(1-z/a_\nu)e^{P_\nu(z/a_\nu)}=R(z)g_1(z),$$
$$h(z)=T(z)e^{v(z)}\prod(1-z/b_\nu)e^{P_\nu(z/b_\nu)}=T(z)h_1(z).$$

Here R and T are polynomials whose degrees are of the same parity and whose highest terms have the same sign; u and v are integral functions; the products are canonical products with real simple zeros. The zeros of one product are separated by those of the other. The Taylor coefficients of R, T, u, v are real. Also $u(z)-v(z)+\sum\{P_\nu(z/a_\nu)-P_\nu(z/b_\nu)\}$ = constant and $h_1'(x)g_1(x)-h_1(x)g_1'(x)>0$ at all points of the real axis with at most one exception. Theorem 2. If $F(z)$ has at most a finite number of zeros in $y<0$, then $F(z)\epsilon B$ if and only if $-i$ is not an asymptotic value of $h(z)/g(z)$. Theorem 3. If $F(z)\epsilon B$ and if $\alpha, \beta, \gamma, \delta$ are real numbers with $\alpha\delta-\beta\gamma>0$, then $TF(z)=(\alpha g(z)+\beta h(z))+i(\gamma g(z)+\delta h(z))\epsilon B$; F and TF have the same number of zeros in $y<0$. Theorem 4. If, for three essentially different systems $\alpha, \beta, \gamma, \delta$, TF has only a finite number of zeros in $y<0$, then $F\epsilon B$. Proofs are outlined; theorem 1 is based on previous results of the author [C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 179–181 (1943); these Rev. **6**, 59]. *W. H. J. Fuchs.*

Meiman, N. On the zeros of linear combinations of a class of entire functions. Doklady Akad. Nauk SSSR (N.S.) **62**, 453–456 (1948). (Russian)

The notation is the same as in the preceding review. The integral functions $g(z)$, $h(z)$ are said to be a "real pair" if $\lambda g+\mu h=0$ has the same number of real roots for all real values λ, μ. It is known that $F=g+ih$ or $F_1=g-ih$ are in the class of functions belonging to B and having no zeros in $y<0$. The author gives best possible upper bounds for the number of complex roots of $f(z)=P(z)g(z)+Q(z)h(z)$, where P, Q are polynomials and g, h form a "real pair." The simplest result is: if P is of degree m, Q of degree n, then $f(z)$ has at most $2[(m+n+1)/2]$ complex roots. The proofs are based on previous work of the author and on the simple lemma: if $u(z)+iv(z)=d(z)F(z)$, where $F\epsilon B$ and d is an integral function taking real values on the real axis, then the number of complex roots of $u(z)=0$ or $v(z)=0$ does not exceed twice the number of roots of $u+iv=0$ in $y<0$. *W. H. J. Fuchs* (Ithaca, N. Y.).

Nehari, Zeev. Analytic functions possessing a positive real part. Duke Math. J. **15**, 1033–1042 (1948).

In the class of functions $f(z)=1+a_1z^{-1}+a_2z^{-2}+\cdots$ which are analytic with positive real part in a region of connectivity n, containing $z=\infty$, it is proved that there exists one which makes $\Re\sum_1^k\gamma_m a_m$ = maximum for given $\gamma_1, \cdots, \gamma_k$ and which maps the region onto a Riemann surface which completely covers the right half-plane n times. This is of the same nature as a recent generalization of Schwarz's lemma by the reviewer [same J. **14**, 1–11 (1947); these Rev. **9**, 24]. The method of proof rests on an auxiliary extremal problem: to find a least coefficient which makes a certain differential nonnegative on the boundary. No attempt is made to interpret this auxiliary problem. The solution of the primary problem is not unique. *L. Ahlfors* (Cambridge, Mass.).

Doss, S. H. On a linear functional. Proc. Math. Phys. Soc. Egypt **3** (1947), 59–62 (1948).

Let S be the complex linear topological space of functions $x(z)$, analytic in $|z|<R$, with convergence meaning uniform convergence in every compact subset. Let $L_n(x)=x^{(n)}(0)/n!$ and $u_n(z)=z^n$. Then, $\{L_n\}$ and $\{u_n\}$ are orthogonal over S, and every $x\epsilon S$ has the representation $x=\sum u_nL_n(x)$. If $\{\lambda_n\}$ is a complex sequence, then a necessary and sufficient condition that $\sum\lambda_nL_n(x)$ converge for every x in S is that $\limsup|\lambda_n|^{1/n}<R$. The author uses this observation to prove that every linear functional F on S has the form $F(x)=\sum\lambda_nL_n(x)$, where $\limsup|\lambda_n|^{1/n}<R$. This is then applied to the uniqueness problem for a basic sequence of polynomials $\{x_k\}$, for which $u_n=\sum\lambda_{nk}x_k$, and $\limsup|\lambda_{nk}|^{1/n}<R$: whenever $x\epsilon S$ has a representation $\sum A_kx_k$, then it is unique. *R. C. Buck* (Providence, R. I.).

Beatty, S. On the minimum value of the Riemann-Roch expression for order-bases in the large. Trans. Roy. Soc. Canada. Sect. III. (3) **42**, 11–13 (1948).

As a consequence to the Riemann-Roch theorem the author determines the minimum value of the expression $N((\tau))+\frac{1}{2}\sum\sum\nu\tau$, where $N((\tau))$ is the dimension of the linear space of functions on a (relative) Riemann surface which have at given places at least the orders

$$(\tau)=\{\tau_1=n_1/\nu_1, \cdots, \tau_r=n_r/\nu_r\},$$

where the ν_i denote the orders of the branches. *O. F. G. Schilling* (Chicago, Ill.).

Beatty, S. On the number of conditions to apply to a function $R(Z, U)$ to build it on an assigned local order-basis (τ). Trans. Roy. Soc. Canada. Sect. III. (3) **42**, 15–18 (1948).

Suppose that $F(Z, U)=0$ is an algebraic equation of degree N with respect to U, and let (τ) denote a set of multiplicities $\tau_i=n_i/\nu_i$ for the branches of order ν_i over the point $Z=0$. The author determines the number of conditions so that the set of functions $(U^{N-1}Q_{N-1}+\cdots+UQ_1+Q_0)/Z^\lambda$, with polynomials Q_i in Z and sufficiently large λ, contains the set of functions whose local orders are at least equal to the given (τ). *O. F. G. Schilling* (Chicago, Ill.).

Hornich, Hans. Die algebraischen Funktionen, deren Iteration die Identität liefert. Monatsh. Math. **52**, 311–322 (1948).

The author determines by essentially nonanalytic methods the algebraic functions $z_1(z_2)$ of one variable which have period two under the operation of functional substitution. Such functions must be given by a nonconstant non-decomposing polynomial $F(z_1+z_2, -z_1z_2, 1)=0$ arising from a homogeneous polynomial of three variables. The author uses the interpretation of the elements of the fractional linear group by points of the projective 3-space. The natural connection with coverings of degree two is used in order to obtain explicit results. *O. F. G. Schilling* (Chicago, Ill.).

Lavrent'ev, M. A general problem of the theory of quasi-conformal representation of plane regions. Mat. Sbornik N.S. **21**(63), 285–320 (1947). (Russian)

The author establishes the existence of a quasi-conformal mapping by a pair of functions which satisfy a system of nonlinear equations. A system (1) $\phi_k(x, y, u, v, u_x, u_y, v_x, v_y)=0$, $u_x=\partial u/\partial x, \cdots, k=1, 2$, is said to admit a quasi-conformal mapping of a domain D of (x, y)-plane into a domain of the (u, v)-plane if there exists a homeomorphic mapping (2) $u=u(x, y)$, $v=v(x, y)$, where u and v satisfy (1). Let the point $w_0=u_0+iv_0$ correspond to $z_0=x_0+iy_0$. By the inverse of the transformation (3) $u-u_0=u_x(x-x_0)+u_y(y-y_0)$, $v-v_0=v_x(x-x_0)+v_y(y-y_0)$ a square $w_0w_1w_2w_3$ whose side w_0w_1 makes the angle ν with the positive u-axis is mapped into a parallelogram $z_0z_1z_2z_3$ one of whose sides, say z_1z_0, and

the angle formed by z_2z_0 and z_1z_0 are $V_\nu \exp(i\alpha_\nu)$ and θ_ν, respectively. Finally let $W_\nu = 1/(V_\nu\Delta)$, where Δ is the determinant of (3). Equations (1) can be replaced by the system (4) $W_\nu = F_1^{(\nu)}(x, y, u, v, V_\nu, \alpha_\nu)$, $\theta_\nu = F_2^{(\nu)}(x, y, u, v, V_\nu, \alpha_\nu)$ and (2) can be written in the form $y = y(x, v)$, $u = u(x, v)$. In analogy with the terminology used in hydrodynamics, the lines $y = y(x, v)$, v = constant are called "streamlines"; $R = y_v$, $\tau = y_x$ are called the "density" and the "inclination" of the streamlines, respectively. The equations (4) are called the "equations in terms of the characteristics" corresponding to (1). If (1) is the Laplace equation, the quantities $P = \log V$ and α are connected by the Cauchy-Riemann equations. The author shows that, if the determinant of (3) is positive, $P = \log V$ and α satisfy the system (5) $P_v = a_1P_u + a_2\alpha_u + a_3$, $\alpha_v = b_1P_u + b_2\alpha_u + b_3$, where the a_n and b_n are functions of W_ν, V_ν, θ_ν, α_ν, and their derivatives with respect to u and v. Finally the author calls the system (4) strongly elliptic if for every ν the functions $F_k^{(\nu)}$, $k = 1, 2$, are single-valued and differentiable, and there exists a positive constant κ, so that for all values of the arguments in $\kappa \leqq \theta_\nu \leqq \pi - \kappa$, the relation $\partial F_1^{(\nu)}/\partial V_\nu \geqq \kappa > 0$ holds. If the system (4) is strongly elliptic then R and τ will satisfy an elliptic system of equations $R_x = \tau_v$, $R_v = a\tau_x + b\tau_v + c$.

The author proves that under certain general conditions, if the system (4) is strongly elliptic, then the corresponding system (5) is elliptic. He also shows that in this case many properties (in particular the Schwarz-Lindelöf principle and maximum principle) of conformal mappings are preserved. Further he shows that if the system (4) corresponding to (1) is strongly elliptic, and the derivatives of the equation of the boundary curves $y = y_0(x)$, $y = Y(x)$ of the strip D are uniformly bounded, then there exists a quasi-conformal mapping (2) corresponding to (4) which maps D into the strip $h < v < H$. Here the points $\pm\infty$ go into the points $\pm\infty$, respectively. The mapping is unique (within a translation).

S. Bergman (Cambridge, Mass.).

Mellihzon, A. S. On the assignment of monogeneity to quaternions. Doklady Akad. Nauk SSSR (N.S.) **59**, 431–434 (1948). (Russian)

N. M. Kryloff [C. R. (Doklady) Acad. Sci. URSS (N.S.) 55, 787–788 (1947); these Rev. **9**, 233] presented a definition of monogeneity for functions of a quaternion variable. The author shows that [as was pointed out in the review of Kryloff's paper] Kryloff's monogenic functions are all linear.

R. P. Boas, Jr. (Providence, R. I.).

Esteban Carrasco, Luis. Distribution of points on homographic circles. Revista Mat. Hisp.-Amer. (4) **8**, 134–142 (1948). (Spanish)

In connection with the study of the derivative of one polygenic function with respect to another, Kasner found it convenient to develop the geometry of homographic circles or clocks. [See Kasner, Atti Congresso Internaz. dei Mat., Bologna, 1928, v. 3, pp. 255–260 (1930); L. Hoffman and Kasner, Bull. Amer. Math. Soc. **34**, 495–503 (1928).] The author discusses theorems given by Kasner and Hoffman, and gives other proofs. *J. De Cicco* (Chicago, Ill.).

Theory of Series

Yü, S. H. Uniform convergence of the amplitude series. Sci. Rep. Nat. Tsing Hua Univ. **5**, 18–28 (1948).

Author's abstract: "A new type of infinite series of variable terms is formulated and is called the amplitude series. Its uniform convergence is proved for the interval of the independent variable equal to any rational real number, arbitrary but finite." The series arose in the author's work on X-rays. The terms of the series involve auxiliary functions involving finite trigonometric sums involving many parameters, and are too complicated for reproduction here. There are misprints. *R. P. Agnew* (Ithaca, N. Y.).

Kreis, H. Über die Summationsformel von Euler. Mitt. Verein. Schweiz. Versich.-Math. **48**, 37–42 (1948).

Fort, Tomlinson. Quasi-monotone series. Amer. J. Math. **71**, 227–230 (1949).

The author considers properties of a certain class of series, which are "quasi-monotone in the mean." *O. Szász.*

Cesco, R. P. On strong summability. Univ. Nac. La Plata. Publ. Fac. Ci. Fisicomat. No. 195, Vol. **4**, núm. 2. Serie Segunda, 17, Revista, 170–178 (1948). (Spanish)

Let A be a regular summability matrix with real nonnegative elements a_{nk}, and let $p > 0$. A sequence s_n of complex numbers is strongly summable to s if $\sum_{k=0}^{\infty} a_{nk}|s_k - s|^p \to 0$ as $n \to \infty$. It is shown that if s_n and t_n are strongly summable to s and t, respectively, and s_n is bounded, then the product sequence s_nt_n is strongly summable to the product st. An example shows that the hypothesis that s_n is bounded cannot be removed. Strong summabilities of different orders are compared with each other and with ordinary summability. The obvious fact, that A strongly includes B if there is a constant K such that $0 \leqq a_{nk} \leqq Kb_{nk}$, is shown to give known inclusion relations among familiar methods. In order that at least one divergent sequence s_n be strongly summable, it is necessary and sufficient that there exist a sequence $k(1), k(2), k(3), \cdots$ of integers such that $\sum_{p=1}^{\infty} a_{n,k(p)} \to 0$ as $n \to 0$. *R. P. Agnew* (Ithaca, N. Y.).

Vermes, P. The application of γ-matrices to Taylor series. Proc. Edinburgh Math. Soc. (2) **8**, 43–49 (1948).

The author extends his previous results [same vol., 1–13 (1947); these Rev. **9**, 234] on series-to-sequence transformations by semi-regular γ-matrices. *G. Piranian.*

Andrianov, S. N. On the strength of methods of summability of series defined by Professor Obreschkoff. Učenye Zapiski Kazan. Univ. **101**, kn. 3, 24–31 (1941). (Russian)

The author seems to be unaware that the methods are the Nörlund methods that have been extensively studied by many authors. *R. P. Agnew* (Ithaca, N. Y.).

Agnew, Ralph Palmer. Abel transforms and partial sums of Tauberian series. Ann. of Math. (2) **50**, 110–117 (1949).

Let $u_1, u_2, \cdots$ be a sequence of complex numbers satisfying $\limsup_{n\to\infty} |nu_n| < \infty$ and $\sigma(t) = \sum_{n=1}^{\infty} u_n t^n$ and $s(n) = \sum_{k=1}^{n} u_k$. It is shown that if $q > 0$, then there exist absolute constants $A = A(q)$ satisfying

$$(1) \quad \limsup_{t\to 1-0} |\sigma(t) - s([q/\log t^{-1}])| \leqq A(q) \limsup_{n\to\infty} |nu_n|$$

for all such sequences $u_1, u_2, \cdots$; in fact, (1) holds if $A(q) = \gamma + \log q + 2\int_q^{\infty} x^{-1}e^{-x}dx$ and for this value of $A(q)$, the equality sign holds in (1) for some sequences $u_1, u_2, \cdots$. The case $q = \log 2$ goes back to Agnew [Duke Math. J. **12**, 27–36 (1945); these Rev. **7**, 12] and to Hadwiger [Revista Mat. Hisp.-Amer. (4) **7**, 65–69 (1947); these Rev. **9**, 86]; the existence of $A(q)$ for $q = 1$ is a consequence of a somewhat

more general result of Wintner [Comment. Math. Helv. **20**, 216–222 (1947); these Rev. **9**, 86] where $|nu_n|$ is replaced on the right of (1) by the quantity $n^{-1}|\sum_{k=1}^{n} ku_k|$, which occurs in Tauber's own theorem; the determination of the best constant $A(1)$ is due to Hartman [Amer. J. Math. **69**, 599–606 (1947); these Rev. **9**, 86] and to Hadwiger [Comment. Math. Helv. **20**, 319–322 (1947); these Rev. **9**, 86]. Agnew's proof of the existence and determination of the best constant $A(q)$ is simple and depends on a modification of regular summation methods to involve inequalities and "lim sup" processes rather than equality and "lim" processes.

P. Hartman (Baltimore, Md.).

Rajagopal, C. T. On some extensions of Ananda Rau's converse of Abel's theorem. J. London Math. Soc. **23**, 38–44 (1948).

The author states that his note is in the nature of an addendum to a note of Bosanquet [same J. **19**, 161–168 (1944); these Rev. **7**, 152]. The extension consists in replacing certain criteria involving sums of blocks of terms of a series by criteria involving certain Riesz means.

N. Levinson (Copenhagen).

Gál, István Sándor, et Koksma, Jurjen Ferdinand. Sur l'ordre de grandeur des fonctions sommables. C. R. Acad. Sci. Paris **227**, 1321–1323 (1948).

A theorem is stated without proof which is asserted to be a generalization of the following Tauberian theorem [cf. Kaczmarz and Steinhaus, Theorie der Orthogonalreihen, Warsaw and Lwów, 1935, p. 8]. Let S be a measurable set, and let $f_n(x)$, $n=1, 2, \cdots$, be in $L^p(S)$, $p\geqq 1$. Let $F(N, x)=\sum_1^N f_n(x)$, and let ϕ be a positive increasing function with $\sum\phi(N)^{-p}<\infty$. Then if $\int_S |F(N, x)|^p dx\leqq\psi(N)$, the relation $F(N, x)=o(\psi(N)^{1/p}\phi(N))$ holds for almost all $x\varepsilon S$. The generalization is too long to be given here.

W. J. LeVeque (Cambridge, Mass.).

Thron, W. J. Twin convergence regions for continued fractions $b_0+K(1/b_n)$. II. Amer. J. Math. **71**, 112–120 (1949).

Regions B_f and B_g are defined as "best" twin convergence regions for the continued fraction (1) $b_0+K[1/b_n]$ if $\sum|b_n|=\infty$, $b_{2n}\varepsilon B_f$, $b_{2n+1}\varepsilon B_g$, $n\geqq 0$, insure convergence of (1), and if there do not exist twin convergence regions B_f' and B_g' such that $B_f'\supset B_f$, $B_g'\supset B_g$, where either $B_f'\neq B_f$, or $B_g'\neq B_g$, or both. A large class of best twin convergence regions is determined as follows. Let $\alpha(\theta)$ be a continuous function of period 2π which satisfies the conditions $|\alpha(\theta)|<\frac{1}{2}\pi-\epsilon_1$, $|\alpha(\theta)-\alpha(\phi)|/|\theta-\phi|<1-\epsilon_2$, $\theta\neq\phi$, $\epsilon_1>0$, $\epsilon_2>0$. Let $f(\theta)=f_0\exp\int_{\pi/2}^{\theta}\tan\alpha(\psi)d\psi$, $f_0>0$. Then the regions B_f and B_g defined by $re^{i\theta}\varepsilon B_f$ if $r\geqq f(\theta)$, $re^{i\theta}\varepsilon B_g$ if $r\geqq 4/f(\pi-\theta)$ are best twin convergence regions for (1). These regions are improvements of certain regions obtained by the author [part I, same J. **66**, 428–438 (1944); these Rev. **6**, 210]. An example shows that an analogous improvement of Van Vleck's criterion is not possible.

E. Frank (Chicago, Ill.).

Fourier Series and Generalizations, Integral Transforms

Bellman, Richard. Some properties of summation kernels. Duke Math. J. **15**, 1013–1019 (1948).

The author proves three theorems. Two are generalizations of a lemma of Paley [Proc. London Math. Soc. (2) **31**, 289–300 (1930), p. 292, lemma 3]. For the Poisson kernel $P(t, \theta)=1+2\sum_{n=1}^{\infty}e^{-nt}\cos n\theta$ Paley's lemma is:

$$P(t_1+t_2, \theta)\leqq c[P(t_1, \theta)+P(t_2, \theta)],$$

where c is independent of t_1, t_2 and θ. The author proves similar inequalities for the cases where e^{-nt} is replaced by n^{-t} or $e^{-n^\rho t}$, $0<\rho\leqq 1$, leaving it unsettled for $\rho>1$. The third theorem contains an improvement of a constant in an inequality of B. M. Eversull [Ann. of Math. (2) **24**, 141–166 (1923), p. 160] on triple Fourier series. If $f(x, y, z)$ is defined over the cube $0\leqq x, y, z\leqq\pi$ and is continuous there, with the Fourier expansion $\sum_{l,m,n=1}^{\infty}a_{lmn}\sin lx\sin my\sin nz$, then

$$\max_{x,y,z;0\leqq t<\infty}\left|\sum a_{lmn}e^{-(l^2+m^2+n^2)t}\sin lx\sin my\sin nz\right|\leqq\max_{x,y,z}|f|.$$

This inequality, with a constant greater than one on the right side, was first proved by Eversull.

K. Chandrasekharan (Princeton, N. J.).

Gál, István Sándor. Sur les séries orthogonales $C(1)$-sommable et $\lambda(n)$-lacunaires. C. R. Acad. Sci. Paris **227**, 1140–1142 (1948).

Let $\lambda(n)\geqq 0$ $(n=1, 2, \cdots)$. A sequence $\{c_n\}$ is called $\lambda(n)$-lacunary [see Alexits, Acta Univ. Szeged. Sect. Sci. Math. **11**, 251–253 (1948); these Rev. **10**, 113] if the number of nonzero terms among $c_n, c_{n+1}, \cdots, c_{2n-1}$ is $O(\lambda(n))$. The author states without proof a number of generalizations of results of Alexits [loc. cit.]. In particular, if $\{\varphi_n(x)\}$ is an orthonormal system, (1) condition $\sum c_n^2\log^2\lambda(n)<\infty$ implies the convergence of $\sum c_n\varphi_n(x)$ almost everywhere; (2) the same conclusion holds if $\sum c_n^2(\log\log n)^2<+\infty$ and if c_n is $\lambda(n)=(\log n)^r$-lacunary $(r>0)$.

A. Zygmund.

Zamansky, Marc. Sur l'approximation des fonctions continues. C. R. Acad. Sci. Paris **227**, 1011–1013 (1948).

The author states a number of results about the approximation of continuous functions by trigonometric polynomials. The following theorem deserves attention. A necessary and sufficient condition that the Jackson sums of a function $F(x)$ of period 2π be $O(n^{-2})$ is that (1) $F'(x)$ exist and be continuous, and (2) the approximation of the function $\tilde{F}'(x)$, conjugate to $F'(x)$, by the Fejér means be $O(n^{-1})$.

A. Zygmund (Chicago, Ill.).

Levitan, B. M. Some questions of the theory of almost periodic functions. I. Uspehi Matem. Nauk (N.S.) **2**, no. 5(21), 133–192 (1947). (Russian)

This paper gives a fairly detailed account of the classical theory of almost periodic (a. p.) functions including the generalizations by Stepanoff and Weyl. In the first chapter the author introduces almost periodic functions using the classical definition. He proves that sum, product and uniform limit of almost periodic functions are almost periodic. Next he discusses Bochner's theory of "normal" functions. At this stage the generalizations by Stepanoff and Weyl (S-a. p. and W-a. p. functions) are defined and the existence of the mean value $\mathfrak{M}\{f(x)\}=\lim_{T\to\infty}T^{-1}\int_a^{a+T}f(x)dx$, where the integral converges uniformly in a, is proved for W-a. p. functions and the Fourier series is introduced.

In chapter II the author studies the theory of Fourier integrals and Fourier-Stieltjes integrals as an introduction to the theory of positively definite functions and Bochner's proof of Parseval's formula [S. Bochner, Vorlesungen über Fouriersche Integrale, Akademische Verlagsgesellschaft, Leipzig, 1932, pp. 63–82]. He further proves by Weyl's method that an a. p. function can be approximated uniformly with any given accuracy by a finite exponential sum.

In chapter III the relations between translation numbers and Fourier exponents are investigated. Bohr's result that the set of translation numbers of a given function and with a given accuracy is "roughly" identical with the set of solutions of a certain system of inequalities $|\Lambda_\nu\tau| \leqq \delta \pmod{2\pi}$, $\nu=1, \cdots, n$, where $\Lambda_1, \cdots, \Lambda_n$ belong to the module of the function, is proved. The author proves also the Bochner-Fejér summability of the Fourier series and he investigates the relations between almost periodic functions and periodic and limit periodic functions of several variables. The set of integral translation numbers is studied [Besicovitch, Almost Periodic Functions, Cambridge University Press, 1932]. A new proof of the main theorems of almost periodic functions is based on the following result: If $g_n(x)$ is a. p. and $h_n(x)$ is continuous and satisfies the condition $\mathfrak{M}\{h_n(x)e^{i\lambda x}\}=0$ for every real λ and $g_n(x)+h_n(x)$ tends uniformly to a limit function then $g_n(x)$ and $h_n(x)$ will also tend uniformly to limit functions. *H. Tornehave.*

Levitan, B. M. Some questions of the theory of almost periodic functions. II. Uspehi Matem. Nauk (N.S.) 2, no. 6(22), 174–214 (1947). (Russian)

In the first chapter the author gives an account of a generalization of a. p. functions introduced by himself [Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) 15, no. 2, 3–34 (1938)]. Let $f(x)$ be a continuous function. A number $\tau=\tau_f(\epsilon, N)$ is called an (ϵ, N)-translation number of $f(x)$ if $|f(x+\tau)-f(x)|<\epsilon$ when $|x|<N$. The continuous function $f(x)$ is called N-almost periodic (N-a. p.) if the set of (ϵ, N)-translation numbers is relatively dense for all positive values of ϵ and N and, furthermore, the translation numbers satisfy the condition $\tau_f(\epsilon, N)\pm\tau_f(\rho, N)=\tau_f(\delta, N)$, where $\delta=\epsilon+\lambda(\rho)$ and $\lambda(\rho)\rightarrow 0$ when $\rho\rightarrow 0$. The starting point is the following theorem by Marčenko [to be published elsewhere]. To an N-a. p. function $f(x)$ and a positive number N corresponds an a. p. function $\varphi(x)$, such that every $\tau_f(\rho, N)$ is a $\tau_\varphi(\lambda(\rho))$, while every $\tau_\varphi(\epsilon)$ is a $\tau_f(\epsilon, N)$. It follows then from the relations between Fourier exponents and translation numbers of a. p. functions that there exists a certain module of real numbers, the module of $f(x)$, such that the set of numbers $\tau_f(\epsilon, N)$ for given ϵ and N is "roughly" the set of solutions of a system of inequalities $|\Lambda_\nu\tau|<\delta \pmod{2\pi}$, $\nu=1, \cdots, n$, where $\Lambda_1, \cdots, \Lambda_n$ belong to the module.

The mean value of an N-a. p. function is not always uniquely determined [B. Levine and B. Levitan, C. R. (Doklady) Acad. Sci. URSS (N.S.) 22, 539–542 (1939); Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) 17, 109–110 (1940); these Rev. 3, 106], but there exists a sequence $T_1, T_2, \cdots$ of positive numbers such that the limit

$$A_k=\lim_{n\to\infty}\frac{1}{2T_n}\int_{-T_n}^{T_n} f(x)e^{-i\mu_k x}dx$$

exists for every μ_k in the module. The series $f(x)\sim\sum A_k e^{i\mu_k x}$ is the Fourier series of $f(x)$. It is not uniquely determined by $f(x)$, even if $f(x)$ is bounded, but if either

$$\limsup_{T\to\infty}\,(2T)^{-1}\int_{-T}^{T}|f(x)|^2dx$$

or

$$\limsup_{T\to\infty}\,(2T)^{-1}\int_{-T}^{T}|f(x)|dx$$

is finite, then the identity theorem nevertheless holds in the sense that $f(x)$ is uniquely determined by any one of its Fourier series. The author gives three proofs of this theorem. The first proof depends on the properties of integral translation numbers. The second proof [by Marčenko] is closely related to the methods applied by Bohr [Fastperiodische Funktionen, Springer, Berlin, 1932]. The third proof is based on Bochner's theory of positively definite functions. The author mentions that a proof by Bogoliuboff [Ann. Chaire Phys. Math. Kiev 4, 185–205 (1939); these Rev. 8, 512] of the identity theorem of a. p. functions could easily be generalized to N-a. p. functions.

Next the author introduces so-called $\tilde{W}$-a. p. functions: $f(x)$ is $\tilde{W}$-a. p. if to $f(x)$ there correspond l and L such that every interval of length l contains a number τ satisfying

$$\sup_{-\infty<\lambda<\infty}\ \sup_{-\infty<s<\infty}\left|L^{-1}\int_s^{s+L}|f(x+\tau)-f(x)|e^{-i\lambda x}dx\right|<\epsilon.$$

The mean value exists for $\tilde{W}$-a. p. functions.

In the last chapter the author investigates systems of differential equations $dy_i/dx=\sum_{k=1}^n f_{ik}(x)y_k+g_i(x)$; $i=1, \cdots, n$. The coefficients $f_{ik}(x)$ and $g_i(x)$ are a. p. functions. If the corresponding homogeneous system possesses no bounded solution except $y_i\equiv 0$, then every bounded solution of the inhomogeneous system is N-a. p. If the homogeneous system has no bounded solution for which $|y_1|^2+\cdots+|y_n|^2$ assumes arbitrarily small values, and the inhomogeneous system has a bounded solution, then the particular solution for which $\sup_{-\infty<x<\infty}(|y_1|^2+\cdots+|y_n|^2)$ has the smallest possible value is N-a. p. Using this result the author proves the classical theorems by Favard on differential equations with almost periodic coefficients [Acta Math. 51, 31–81 (1927)]. *H. Tornehave* (St. Johns, Que.).

Solodovnikov, V. V. Criteria for the absence of overregulation and criteria for monotonicity. Doklady Akad. Nauk SSSR (N.S.) 62, 599–602 (1948). (Russian)

The physical problem reduces to the mathematical problem of determining $f(s)$ so that $\int_0^\infty f(s)\cos st\,ds>0$ for $t>0$. This problem has been treated by Mathias [Math. Z. 16, 103–125 (1923)] and by Boas and Kac [Duke Math. J. 12, 189–206 (1945); these Rev. 6, 265]. The author discusses some simple consequences of the results of these authors.

R. Bellman (Stanford University, Calif.).

van Dantzig, D. On the inversion of k-dimensional Fourier-Stieltjes-integrals. Nederl. Akad. Wetensch., Proc. 51, 858–867 = Indagationes Math. 10, 286–295 (1948).

Let $V(X)$ be a completely additive set function defined on all Borel sets X in R_k, and let

$$\phi_V(t)=\int_{R_k}\exp(ix\cdot t)dV(x), \qquad x\cdot t=\sum_{r=1}^k x_r t_r.$$

Defining

$$W_\delta(A)=\delta^{-k}\int_A\exp(-i\delta^{-1}x\cdot t)dx\int_{R_k}\phi_V(\delta^{-1}t)dH(t),$$

where A is an interval in R_k and $H(X)$ is completely additive, the author proves that $V(A)=\lim_{\delta\to 0}W_\delta(A)$ provided that V is continuous on the frontier of A and H satisfies certain conditions (expressed in terms of $\phi_H(t)$) too lengthy to quote. If H is absolutely continuous, $H(X)=\int_X h(t)dt$, the conditions on H are satisfied provided that $h(t)$ is continuous, of bounded variation, summable over R_k, even in each coordinate t_r, and $h(0)=(2\pi)^{-k}$. Special cases are: (1) $h(t)=(2\pi)^{-k}$ if $|t_r|<1$ $(r=1, \cdots, k)$, 0 elsewhere; this gives the known inversion formula [Lévy, Haviland; cf. H.

Cramér, Random Variables and Probability Distributions, Cambridge University Press, 1937]:

$$V(A)=\lim_{T\to\infty}(2\pi)^{-k}\int_{|t_r|<T}\phi_V(t)dt\int_A \exp(-ix\cdot t)dx;$$

(2) $h(t)=(2\pi)^{-k}\exp\{-\frac{1}{2}Q(t)\}$, $Q(t)$ being a positive definite quadratic form in $t_1, \cdots, t_k$; this gives

$$V(A)=\lim_{\delta\to 0}(2\pi)^{-k}\int_{R_k}\phi_V(t)\exp(-\tfrac{1}{2}\delta^2Q(t))dt\int_A\exp(-ix\cdot t)dx$$

and this inversion formula holds for any domain A containing no discontinuities of V on its frontier. The paper also contains explicit bounds for the error $|W_\delta(A)-V(A)|$.

G. E. H. Reuter (Manchester).

Akutowicz, Edwin J. The third iterate of the Laplace transform. Duke Math. J. **15**, 1093–1132 (1948).

The author constructs a representation and inversion theory for the integral equation (*) $f(x)=\int_{0+}^\infty e^{xt}E(xt)\phi(t)dt$ which is the Laplace transform three times iterated. Here $E(x)$ is the exponential integral $\int_x^\infty e^{-t}t^{-1}dt$. It is shown that if

$$(**)\quad B_{k,t}[f(x)]=\frac{-k}{(k!)^2(k-1)!}\left(\frac{k}{t}\right)^{k+1}[x^{2k-1}f^{(k-1)}(x)]^{(2k)}_{x=k/t}$$

then $\lim_{k\to\infty}B_{k,t}[f(x)]=\phi(t)$, a formula also obtained, though differently, by H. Pollard [same J. **14**, 659–674 (1947); these Rev. **9**, 237]. Necessary and sufficient conditions for a function $f(x)\epsilon C_\infty$, $0<x<\infty$, to be representable in the form (*) with $\phi(t)$ belonging to various special function classes are obtained in terms of the behavior of the $B_{k,t}[f(x)]$, $k=1, 2, \cdots$.

I. I. Hirschman, Jr.

Cherry, T. M. Expansions in terms of parabolic cylinder functions. Proc. Edinburgh Math. Soc. (2) **8**, 50–65 (1948).

The author proves that if $f(x)$ is of bounded variation in every finite interval and is absolutely integrable in $-\infty<x<\infty$ then

$$-4\pi i f(x)=\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\frac{e^{\frac{1}{2}(\nu+\frac{1}{2})\pi i}}{\sin\nu\pi}d\nu\int_{-\infty}^{\infty}f(t)K(\nu,x,t)dt,$$

where

$$K(\nu,x,t)=D_\nu(x\eta)D_{-\nu-1}(t\bar\eta)+D_\nu(-x\eta)D_{-\nu-1}(-t\bar\eta),$$
$$\eta=e^{i\pi/4},\quad \bar\eta=e^{-i\pi/4}.$$

An equivalent formula was investigated earlier by W. Magnus [Jber. Deutsch. Math. Verein. **50**, 140–161 (1940); these Rev. **2**, 56] under much more restrictive conditions.

I. I. Hirschman, Jr. (Cambridge, Mass.).

Miles, John W. On vector transforms. Physical Rev. (2) **74**, 1531 (1948).

Solutions of the scalar wave equation are often represented as superposition of waves of a given type (for instance, plane or cylindrical waves), and such representations lead to integral transforms (for instance, to Fourier and Fourier-Bessel transforms). The author states [without proof] the corresponding formulae for vector solutions of the wave equation.

A. Erdélyi (Edinburgh).

Bose, N. N. A theorem in operational calculus. Philos. Mag. (7) **39**, 821–823 (1948).

If $f(p)=p\int_0^\infty e^{-px}h(x)dx$ then $\int_1^\infty f(p)\phi(p)dp=\int_0^\infty h(x)\psi(x)dx$, where $\psi(x)=\int_1^\infty pe^{-px}\phi(p)dp$ [the definition of ψ in the paper seems to contain a misprint]. Examples of this transformation are given: they involve functions of the hypergeometric type.

A. Erdélyi (Edinburgh).

van der Pol, Balth., and Bremmer, H. Modern operational calculus based on the two-sided Laplace integral. I. Nederl. Akad. Wetensch., Proc. **51**, 1005–1012 = Indagationes Math. **10**, 338–345 (1948).

van der Pol, Balth., and Bremmer, H. Modern operational calculus based on the two-sided Laplace integral. II. Nederl. Akad. Wetensch., Proc. **51**, 1125–1136 = Indagationes Math. **10**, 349–360 (1948).

The principal operational properties of the integral transformation

$$f(p)=p\int_{-\infty}^{\infty}e^{-pt}h(t)dt$$

of functions $h(t)$ are noted. Since the integral here reduces to the ordinary Laplace transform in case $h(t)=0$ when $t<0$, the two-sided transformation includes the ordinary one as a special case. The authors point out advantages of the operational calculus based on this less common transformation. Some of the operational properties are simpler than the corresponding ones for the Laplace transformation. Broader classes of object functions $h(t)$ and result functions $f(p)$ can be included and consequently broader applications to properties of special functions are possible. The examples cited here in linear ordinary differential equations, however, display no advantage of simplicity or clarity in such applications.

R. V. Churchill (Ann Arbor, Mich.).

Ditkin, V. A. Operational calculus. Uspehi Matem. Nauk (N.S.) **2**, no. 6(22), 72–158 (1947). (Russian)

Considerable space is devoted in the first four sections to properties of the Laplace integral, $L\{f\}=\int_0^\infty e^{-pt}f(t)dt$. Following this, the operator $F(D)$ is defined by the relation $F(D)f(t)=g(t)$, where $F(p)L\{f\}=L\{g\}$. This definition is framed so that $D^nf(t)$ (n a positive integer) has meaning only if $f(t)$ and its first $n-1$ derivatives vanish at $t=+0$. Nonvanishing initial conditions are brought in by writing, for example, $D[f(t)-f(+0)]=f'(t)$. Through properties of the Laplace integral, with the inversion integral, Duhamel's integral, series and residue theory, other operators are developed. Then, after a section on application of operators when a parameter is involved, the solution of differential equations and boundary value problems is explained. Four examples in the last section illustrate the procedure. An extensive list of operators, applied to the Heaviside unit step function, is appended.

R. E. Gaskell (Ames, Iowa).

Bouthillon, L. Oscillations et phénomènes transitoires. Leur étude par les transformations de Laplace et de Cauchy. Ann. Radioélec. **2**, 287–328 (1947).

This exposition includes a short table of transforms.

Polynomials, Polynomial Approximations

Horváth, Jean. Sur un théorème de M. Mandelbrojt concernant l'approximation polynomiale des fonctions sur tout l'axe réel. C. R. Acad. Sci. Paris **227**, 889–891 (1948).

The author derives, in a more general form, results of Mandelbrojt on weighted approximation on $(-\infty, \infty)$ by

polynomials $\sum a_n x^{\lambda_n}$ [same C. R. 226, 1668–1670 (1948); these Rev. 9, 583]. *R. P. Boas, Jr.* (Providence, R. I.).

Horváth, Jean. Sur l'approximation polynomiale des fonctions sur une demi-droite. C. R. Acad. Sci. Paris 227, 1074–1076 (1948).

The author derives from a theorem of Mandelbrojt and MacLane [Trans. Amer. Math. Soc. 61, 454–467 (1947); these Rev. 8, 508] the following result. Let $F(x)\geqq 0$, $0\leqq x<\infty$; let $\log F(x)$ be a concave function of $\log x$ and let $\int^\infty x^{-\frac{3}{2}}\log F(x)dx=\infty$. Let $f(x)$ be continuous in $(0,\infty)$ with $\lim_{x\to\infty} f(x)=0$. Then $f(x)$ can be uniformly approximated by $P_n(x)/F(x)$, where $P_n(x)$ are polynomials. Corresponding results for L^p approximation and for the Stieltjes moment problem are given, and a result is announced which applies to polynomials $\sum a_n x^{\lambda_n}$ [cf. the preceding review]. [For a similar problem on $(-\infty,\infty)$, cf. Ahiezer and Babenko, Doklady Akad. Nauk SSSR (N.S.) 57, 315–318 (1947); these Rev. 9, 141.] *R. P. Boas, Jr.*

Valiron, Georges. Remarque sur la représentation approchée par des polynomes des fonctions continues de plusieurs variables. Bull. Sci. Math. (2) 72, 9–12 (1948).

An elementary discussion of the Weierstrass approximation theorem in several variables. *H. Whitney.*

Gavurin, M. K. On linear differential equations with singularities of the second order. Doklady Akad. Nauk SSSR (N.S.) 62, 5–8 (1948). (Russian)

This paper continues the author's researches [Izvestiya Akad. Nauk SSSR. Ser. Mat. 12, 15–30 (1948); these Rev. 9, 430] on the completeness of "D-polynomials" $Dp(x)$, where D is a linear differential operator and p a polynomial. Now $D=g_0(x)(d/dx)^2+g_1(x)d/dx+g_2(x)$ $(x\epsilon I=[a,b];\ g_i\epsilon C;$ $E_0=E(g_0=0)$ and $E_1=E(g_1=0)$ are null sets). The novelty lies in the fact that $E_0\cap E_1$ need not be empty. A regular solution of (1) $Dy=A(x)$ $(A\epsilon C)$ is defined by the conditions (a) $y, y', y''\epsilon C(I-E_0)$; (b) $y, y'\epsilon C(I-(E_0\cap E_1))$; (c) if $g_0(x)y''$ is put equal to 0 in E_0, $g_1(x)y'=0$ in $E_0\cap E_1$, then $g_0(x)y''$, $g_1(x)y'\epsilon C(I)$; (d) (1) holds in I. Let $\Gamma(D)$ be the closure in C of all D-polynomials, $\bar\Gamma(D)$ the set of all functions $A(x)$ such that $A=\lim Dp_n(x)$ where each of the three terms of Dp_n converges uniformly to a limit. If $E_0\cap E_1=0$, then $\Gamma(D)=\bar\Gamma(D)$. It is stated that this is no longer true in the general case. It is proved that $A(x)\epsilon\bar\Gamma(D)$ if and only if (1) has a regular solution y such that (i) y' is absolutely continuous in every interval $[\alpha,\beta]$ with $\int_\alpha^\beta dx|g_0(x)|^{-1}<\infty$, (ii) y is absolutely continuous in every interval $[\alpha,\beta]$ in which $\sup|\int_\alpha^\beta u(x)dx|<\infty$, where $u(x)$ is subject to the conditions $u(x)$ is absolutely continuous, $|u(x)|\leqq 1/|g_1(x)|$, $|u'(x)|\leqq 1/|g_0(x)|$ almost everywhere. It is stated without proof that there is an example of an operator D such that (1) has a regular solution for every $A(x)\epsilon C$ and yet $\bar\Gamma(D)\neq C$. This cannot happen unless $E_0\cap E_1\neq 0$ [see loc. cit.]. In the example $E_0\cap E_1$ consists of three points. *W. H. J. Fuchs* (Ithaca, N. Y.).

Geronimus, Ya. L. On asymptotic properties of polynomials deviating least from zero in the space L_σ^p. Doklady Akad. Nauk SSSR (N.S.) 62, 9–12 (1948). (Russian)

The author states without proof a number of results concerning the asymptotic properties of polynomials deviating least from zero, in the metric L_σ^p, on a rectifiable Jordan curve C in the complex plane x. Let $d\sigma(s)$ be a positive mass distribution on C and let $P_n(x)=x^n+\cdots$ be the polynomial of degree n $(=1,2,\cdots)$ minimizing the integral

$$\left\{(2\pi)^{-1}\int_C|\xi^n+\cdots|^p d\sigma(s)\right\}^{1/p}$$
$$\geqq\left\{(2\pi)^{-1}\int_C|P_n(\xi)|^p d\sigma(s)\right\}^{1/p}=h_n.$$

[For $p=2$ and $d\sigma(s)=w(s)ds$, see Szegö [Math. Z. 9, 218–270 (1921); Trans. Amer. Math. Soc. 37, 196–206 (1935); Orthogonal Polynomials, Amer. Math. Soc. Colloquium Publ., v. 23, New York, 1939; these Rev. 1, 14]; see also Ahiezer [Lectures on the Theory of Approximation, Moscow-Leningrad, 1947; these Rev. 10, 33].] Let $x=\varphi(w)=cw+c_0+c_1w^{-1}+\cdots$ be the function mapping conformally the exterior G of C onto $|w|>1$, and let $w=\delta(x)$ be the inverse function. If (*) $\int_C\log\sigma'(s)\delta'(\xi)|d\xi|>-\infty$ then there is a function $\Delta(x)$ regular and nonvanishing in G whose boundary values (along nontangential paths) satisfy at almost every point of C the relation $\sigma'(s)=|\delta'(\xi)|\lim_{x\to\xi}|\Delta(x)|^{-p}$ $(x\epsilon G,\ \xi\epsilon C)$ [see Szegö, loc. cit.]. The author states that (1) the existence of the integral (*) is both necessary and sufficient that the system $\{P_n(\xi)\}_{n=1,2,\cdots}$ be nonclosed in the space L_σ^p of all complex-valued functions $f(\xi)$ with norm $\|f\|=\{\int_C|f(\xi)|^p d\sigma(s)\}^{1/p}$, $p\geqq 1$, $\xi=\xi(s)$. (2) From (1) follows the existence of the limit $\lim_{n\to\infty}h_n/c^n=1/\Delta(\infty)$, and conversely the condition $\limsup h_n/c^n>0$ implies (1). (3) From (1) follow the formulas

$$\lim_{n\to\infty}\int_C\left|\left\{\frac{P_n(\xi)}{c^n\delta^n(\xi)}\right\}^{p/2}-\{\lambda(\xi)\}^{p/2}\right|^2d\sigma(s)=0,$$
$$P_n(x)\simeq h_n\delta^n(x)\Delta(x).$$

Here the second relation holds uniformly in any closed subdomain of G; $\lambda(\xi)=\Delta(\xi)/\Delta(\infty)$ at the points where $\sigma'(s)$ exists and is positive, and $\lambda(\xi)=0$ elsewhere. *A. Zygmund* (Chicago, Ill.).

Special Functions

Pollard, Harry. The completely monotonic character of the Mittag-Leffler function $E_a(-x)$. Bull. Amer. Math. Soc. 54, 1115–1116 (1948).

W. Feller communicated to the author the result that if $E_a(x)$ is the Mittag-Leffler function $E_a(x)=\sum_{k=0}^\infty x^k/\Gamma(ka+1)$, and if $0<a<1$ then $E_a(-x)=\int_0^\infty e^{-xt}\phi(t)dt$ $(-\infty<x<\infty)$ with $\phi(t)\geqq 0$; that is, $E_a(-x)$ is a completely monotonic function. A proof is given of this fact and it is further shown that $\phi(t)$ admits the representation

$$\phi(t)=(\pi a)^{-1}\sum_{k=1}^\infty\frac{(-1)^{k-1}}{k!}\sin\pi ak\,\Gamma(ak+1)t^{k-1}.$$

Essential use is made of the author's paper [same Bull. 52, 908–910 (1946); these Rev. 8, 269]. *I. I. Hirschman, Jr.* (Cambridge, Mass.).

Bailey, W. N. A double integral. J. London Math. Soc. 23, 235–237 (1948).

It is shown that the integral

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{d\theta d\varphi}{(1-k_1^2\sin^2\theta-k_2^2\sin^2\varphi)^{\frac{1}{2}}}$$

can be evaluated in terms of complete elliptic integrals of the first kind. The proof is based on properties of Appell's

generalized hypergeometric functions found earlier by Appell and the author. *Z. Nehari* (St. Louis, Mo.).

Rutgers, J. G. Extension of some identities. I. Nederl. Akad. Wetensch., Proc. **51**, 868–873 = Indagationes Math. **10**, 296–301 (1948). (Dutch)

In an earlier communication [Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde **52**, 163–167 (1943); these Rev. **8**, 155] the author found alternative expressions for

$$\sum_{p=0}^{s} \frac{(\pm 1)^p (2p+\nu)^k}{(s-p)!\Gamma(s+p+\nu+1)}$$

for some selected values of ν and k. In the present instalment the same method is used to find some transformations of the above sum for $(+1)^p$, arbitrary ν, and a positive even integer k. *A. Erdélyi* (Edinburgh).

Bose, N. N. On some integrals involving E-functions. Philos. Mag. (7) **39**, 824–826 (1948).

Two integrals are evaluated by the operational calculus. The first appears to be a paraphrase, in MacRobert's notation, of a known integral [equation (2.06) of reference (7) of the paper]; the other is

$$\int_0^\infty e^{-y}(xy)^{-\frac{1}{2}\nu} J_\nu(2\sqrt{(xy)})y^{\gamma-1}E(\alpha, \beta::y)dy,$$

whose value is expressed as an infinite series of hypergeometric functions of the type ${}_2F_2$. *A. Erdélyi.*

Inui, Teturo. Unified theory of recurrence formulas. I. Progress Theoret. Physics **3**, 168–187 (1948).

The author lists 18 differential operators of the form $z^a(z-1)^b(d/dz)z^h(z-1)^k$ which will change Riemann's P-scheme into a contiguous scheme ("stair-operators"). These operators are used to derive recurrence relations for the hypergeometric functions and for Legendre functions. *A. Erdélyi* (Edinburgh).

Busbridge, Ida W. Some integrals involving Hermite polynomials. J. London Math. Soc. **23**, 135–141 (1948).

A general result concerning

$$I^a_{mn\cdots} = \int_{-\infty}^\infty e^{-x^2/a}H_m(x)H_n(x)\cdots dx \quad (a>0)$$

and evaluation of I^a_{mn} and of I^a_{mnp}. [Reviewer's remark: similar integrals have been evaluated by K. Mayr, Math. Z. **39**, 597–604 (1935); and the reviewer, Math. Z. **40**, 693–702 (1936).] *A. Erdélyi* (Edinburgh).

Harmonic Functions, Potential Theory

Verblunsky, S. A note on positive harmonic functions. Proc. Cambridge Philos. Soc. **44**, 289–291 (1948).

If $h(x, y)$ is a positive harmonic function in the upper half plane $y>0$, then there are a nonnegative constant k and a bounded nondecreasing function g such that

$$(1) \qquad h(x, y) = ky + \pi^{-1}\int_{-\infty}^\infty \frac{y(1+\xi^2)}{y^2+(x-\xi)^2}dg(\xi).$$

Consider for $\lambda>1$ the transformation

$$Re^{i\Omega} = X+iY = (x+iy)^\lambda = (re^{i\omega})^\lambda,$$

mapping the sector $0<\omega<\pi/\lambda$ into the upper half plane $Y>0$ in one-to-one conformal fashion. Thus the function H defined by the equation $H(X, Y) = h(x, y)$ is harmonic and positive in this upper half plane, and hence is expressible in the form (1) with capital letters replacing small letters. The problem is to find the connection between the functions g and G. This is accomplished for functions g behaving suitably near $\xi=0$ by proving the following theorem. If $0\leqq\delta<2$ and $a>0$, then either $\int_0^a hr^{-\delta}dr$ is infinite for all ω in $0<\omega<\pi$ or is finite for all such ω. In this latter case, if $\xi=a$ is a point of continuity of g, then

$$\lim_{\omega\to 0}\int_0^a hr^{-\delta}dr = \int_0^a (1+\xi^2)r^{-\delta}dg(\xi),$$

both sides being finite. *W. Gustin* (Bloomington, Ind.).

Górski, Jerzy. Sur l'équivalence de deux constructions de la fonction de Green généralisée d'un domaine plan quelconque. Ann. Soc. Polon. Math. **21**, 70–73 (1948).

Considérons dans la plan un domaine infini D de frontière F bornée (F de diamètre transfini non nul). Si les ζ_i ($i=0, 1, \cdots, n$) sont des points distincts variables sur F, le produit $\prod|\zeta_j-\zeta_k|$, $0\leqq j<k\leqq n$, a un maximum obteint pour un système de points η_i. Notons les dans un ordre tel que parmi les $\Delta_j = |\eta_j-\eta_0|\cdots|\eta_j-\eta_{j-1}|\,|\eta_j-\eta_{j+1}|\cdots|\eta_j-\eta_n|$, Δ_0 soit minimum et formons

$$L_n(z) = (z-\eta_1)\cdots(z-\eta_n)/(\eta_0-\eta_1)\cdots(\eta_0-\eta_n).$$

L'auteur montre que $n^{-1}\log|L_n|$ tend vers la fonction de Green de D, pour la position du pôle à l'infini. Il fait pour cela un passage à la limite très simple (par approximation du domaine) en s'appuyant sur ce que la proposition est conséquence immédiate de résultats connus [Leja, mêmes Ann. **12**, 57–71 (1934)] lorsque F satisfait à certaines conditions restrictives (qui écartent d'ailleurs dans le plan les points irréguliers).

Mais je ferai remarquer que le théorème général peut s'obtenir directement et brièvement comme suit. D'abord il est à peu près connu [voir Frostman, Thèse, Lund, 1935, où est explicité p. 46 le cas analogue de l'espace] que la distribution des masses ponctuelles $1/n$ aux points η_i (avec ou sans η_0) converge vaguement (au sens de H. Cartan) vers la distribution d'équilibre de la masse 1 sur F (celle qui donne un potentiel logarithmique borné et égal quasi-partout à une constante sur F); de sorte que $n^{-1}\sum_i \log(1/|z-\eta_i|)$, $i=1, \cdots, n$, tend sur D vers le potentiel de cette distribution. D'autre part $n^{-1}\log\Delta_0^{-1}$ est compris entre

$$\frac{2}{n(n+1)}\sum_{\mu<\nu}^{0,1,\cdots,n}\log\frac{1}{|\eta_\mu-\eta_\nu|}$$

ou

$$\frac{2}{n(n+1)}\min_{\zeta_i}\sum_{\mu<\nu}^{0,1,\cdots,n}\log\frac{1}{|\zeta_\mu-\zeta_\nu|}$$

et

$$n^{-1}\max_{(\zeta_\mu)}\min_{(\zeta\text{ sur }F)}\sum_{\mu=0}^{n}\log(1/|\zeta-\zeta_\mu|).$$

Les termes extrêmes tendent, d'après la théorie du diamètre transfini, vers la constante de la distribution d'équilibre d'où le théorème en vue. Ce raisonnement s'étend d'ailleurs aussitôt aux espaces à n dimensions en donnant une convergence vers la fonction de Green de pôle à l'infini [voir sur ce sens, Brelot, Ann. Sci. École Norm. Sup. (3) **61**, 301–332 (1945); ces Rev. **7**, 204]. *M. Brelot* (Grenoble).

Korovkin, P. P. On the growth of polynomials on a set. Doklady Akad. Nauk SSSR (N.S.) 61, 781–784 (1948). (Russian)

Let E be a point set in the complex plane, and $\{P_n(z)\}$, $n=1, 2, \cdots$, a sequence of polynomials with P_n of degree n, such that for every $z\epsilon E$, (1) $\limsup_{n\to\infty} |P_n(z)|^{1/n} \leqq 1$. Set $l(E) = \limsup_{n\to\infty} \{\max |P_n(z)|\}^{1/n}$ ($z\epsilon \bar{E}$ = closure of E); $l(E)$ depends on $\{P_n(z)\}$. For all $\{P_n(z)\}$ satisfying (1), let $L(E) = \sup l(E)$. Theorems concerning $L(E)$ are proved. First some simple results are stated: $L(E) \geqq 1$; $L(E)=1$ if E consists of a finite number of points; $L(E)=\infty$ if E is unbounded; if from E there is removed a finite number of isolated points E_1, then $L(E-E_1)=L(E)$; $L(E_1+E_2) \leqq \max \{L(E_1), L(E_2)\}$; if $E_1 \subset E_2$ and $\bar{E}_1 = \bar{E}_2$, then $L(E_1) \geqq L(E_2)$. Let F_σ be the sum of a countable number of closed sets. It is proved that (2) $L(E) = \sup L(F_\sigma)$ for all F_σ with $E \subset F_\sigma \subset \bar{E}$. (This is done by showing separately that (3) $L(E) \geqq \sup L(F_\sigma)$ and (4) $L(E) \leqq \sup (F_\sigma)$.)

Let F be a bounded closed set of positive capacity $\tau(F)$, and let $D(F)$ be that one of the regions complementary to F that contains the point $z=\infty$. There is a Green's function $g_F(z)$ for $D(F)$, harmonic everywhere in $D(F)$ except at $z=\infty$, in the neighborhood of which $g_F(z) = \log |z| + u(z)$, with $u(z)$ harmonic in neighborhood of $z=\infty$, and $\tau(F) = e^{-u(\infty)}$. Let (5) $E = \sum_1^\infty F_n$ ($F_1 \subset F_2 \subset \cdots$), $\tau(F_1) > 0$. Then $\{g_{F_n}(z)\}$ decreases monotonically in the region $D(\bar{E})$ to a function (6) $g_F(z) = \lim_{n\to\infty} g_{F_n}(z)$. Some properties: $g_E(z)$ is independent of the particular sequence $\{F_n\}$ used in (5); $E_1 \subset E_2$ implies $g_{E_1}(z) \geqq g_{E_2}(z)$ in $D(\bar{E}_2)$; if $E_1 \subset E_2$, then $g_{E_1}(z) = g_{E_2}(z)$ if and only if $\tau_*(E_1) = \tau_*(E_2)$ [using the notation in Korovkin, same Doklady (N.S.) 58, 1589–1591 (1947); these Rev. 9, 339]; $g_E(z) = g_{\bar{E}}(z)$ if and only if E is τ-measurable.

Let Γ be the boundary of the region $D(\bar{E})$ and Γ' the set of limit points of Γ. Let $\alpha(x) = \limsup_{z\to x} g_E(z)$ for $x\epsilon\Gamma$, $z\epsilon D$, and set $\alpha(E) = \max \alpha(x)$, $x\epsilon\Gamma'$. Then $\alpha(E) \leqq \log d(E)/\tau_*(E)$, where $d(E)$ is the diameter of E. If $t_n(z; F)$ is that polynomial of degree n with leading coefficient unity, having its roots in the set F, that deviates least from zero on F, and if $m_n(F) = \max |t_n(z; F)|$ ($z\epsilon F$), it can be shown that $g_F(z) = \lim n^{-1} |t_n(z; F)/m_n(F)|$, $z\epsilon D(F)$. It is shown [the proofs are too lengthy to sketch] that $L(E) \leqq e^{\alpha(E)}$, $L(E) \geqq e^{\alpha(E)}$, so that $L(E) = e^{\alpha(E)}$. In consequence, if E is a bounded set with positive capacity $\tau(E)$, then $L(E) \leqq d(E)/\tau_*(E) < \infty$. If E is an infinite F_σ-set and $\tau_*(E) = 0$, then $L(E) = \infty$.

Let D be a region containing the point $z=\infty$, and let Γ be its boundary. Using the definition that the Dirichlet problem is said to have a solution in D if to every function $f(x)$ continuous on Γ there corresponds a function $u_f(z)$ harmonic in D and having the limit value $f(x)$ for points x on the boundary, then it can be shown that: the Dirichlet problem has a solution in D if and only if (i) Γ has no isolated points, and (ii) $L(\Gamma) = 1$. *I. M. Sheffer.*

Šerman, D. I. On certain spatial problems of potential theory. Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 329–338 (1948). (Russian)

Let V be a bounded convex body whose surface S has continuous curvature. It is required to determine a function $u(x, y, z)$ harmonic in V, having continuous partial derivatives of orders $1, 2, \cdots, m$ in $V+S$, and satisfying

$$\sum_{k=0}^{m} \sum_{l=0}^{k} \sum_{j=0}^{l} a_{k-l,\, l-j,\, j} \frac{\partial^k u}{\partial x^{k-l} \partial y^{l-j} \partial z^j} = f(x, y, z)$$

on S, where f and the a_{klj} are given functions with $|a_{klj}(P_1) - a_{klj}(P_2)| < K\overline{P_1P_2}$. Assuming that there is a solution of the form $u = \int_S \nu F(x, y, z; \xi, \eta, \zeta) dS$, where F is a certain given function of the a_{klj}, and ν is the unknown density function, the author solves the problem by reducing it to a Fredholm integral equation for ν.

E. F. Beckenbach (Los Angeles, Calif.).

Dubošin, G. N. Expansion of the potential of the ring, the disk and the spheroid. Vestnik Moskov. Univ. **1948**, no. 1, 53–65 (1948). (Russian)

The reciprocal distance

$$\Delta = \{(z-\bar{z})^2 + \rho^2 + \bar{\rho}^2 - 2\rho\bar{\rho}\cos(v-\bar{v})\}^{\frac{1}{2}}$$

between two points with cylindrical coordinates (z, ρ, v) and $(\bar{z}, \bar{\rho}, \bar{v})$ is expanded in a series

$$1/\Delta = \sum_{n=0}^{\infty} Z_{2n}(z-\bar{z})^{2n}, \quad |z-\bar{z}| < \Delta_0 = \{\rho^2 + \bar{\rho}^2 - 2\rho\bar{\rho}\cos(v-\bar{v})\}^{\frac{1}{2}},$$

$$Z_{2n} = (P_{2n}/\bar{\rho}^{2n+1}) \sum_{\nu=0}^{\infty} G_\nu^{2n}(\gamma)(\rho/\bar{\rho})^\nu, \quad \rho < \bar{\rho};$$

$$Z_{2n} = (P_{2n}/\rho^{2n+1}) \sum_{\nu=0}^{\infty} G_\nu^{2n}(\gamma)(\bar{\rho}/\rho)^\nu, \quad \rho > \bar{\rho},$$

where $G_0^{2n}(\gamma)$ are the generalized Legendre (Gegenbauer) polynomials in $\gamma = \cos(v-\bar{v})$ and P_{2n} is a numerical factor. By integrating $f\sigma/\Delta$ (σ, the density) the author finds double power series in z and ρ or z and $1/\rho$ for the potential of a heavy homogeneous circular line, a homogeneous circular flat ring and a homogeneous circular disk. When R_1 and R_2 are the inner and outer radii of the ring, the first expansion converges when $0 \leqq \rho < R_1$, $|z| < R_1 - \rho$; the second when $R_2 < \rho < \infty$, $|z| < \rho - R_2$. The expansions remain convergent when $\rho = R_1$ or $\rho = R_2$, respectively. This leads to expansions valid at inner and boundary points of the ring and disk. A double power series in $1/z$ and ρ valid when $|z| > \rho + R_2$ is mentioned without proof. Next the author considers the potential of a nonhomogeneous oblate spheroid $\bar{x}^2/a^2 + \bar{y}^2/a^2 + \bar{z}^2/c^2 = 1$ ($a > c$) and finds a double power series in z and $1/\rho$ valid when $a \leqq \rho < \infty$, $|z| < \rho - a$ and specializes his result for the case that the density is constant on concentric spheroids $(\bar{x}^2 + \bar{y}^2)/a^2 + \bar{z}^2/c^2 = b^2$ ($0 \leqq b \leqq 1$). An expansion in $1/z$ and ρ valid when $|z| > \rho - a$ is again mentioned without proof. *W. H. Muller.*

Miranda, C. Sull'approssimazione delle funzioni armoniche in tre variabili. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 4, 530–533 (1948).

A new proof, using functional analysis, is given of the possibility of approximating functions of three variables harmonic in a bounded domain and continuous on the corresponding closed set by linear combinations of spherical harmonics and functions derivable from these by a reciprocal radius transformation. *E. N. Nilson* (Hartford, Conn.).

Biernacki, Mieczysław. Sur les valeurs moyennes des fonctions sousharmoniques. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. 1, 13–17 (1946). (French. Polish summary)

Define $I(r, u) = (2\pi)^{-1} \int_0^{2\pi} u(r, \theta) d\theta$ and

$$\phi(r, u) = (2\pi)^{-1} \int_0^{2\pi} \max_{0 \leq t \leq r} u(t, \theta) d\theta.$$

The author establishes the following inequalities for con-

tinuous, nonnegative and subharmonic functions:

$$\phi(r, u) \leqq \frac{1}{\pi}\left[\log\frac{R+r}{R-r}+\frac{5}{2}\right]I(r, u),$$

$$\phi(r, u) < AI(r, u)+\frac{B}{2\pi}\int_0^{2\pi} u(r, \theta)\log^+ u(r, \theta)d\theta,$$

$$\phi(r, u^p) \leqq 2CI(r, u^p), \quad p>1.$$

He reduces the problem to the case of a harmonic function and utilizes the Poisson integral and certain theorems by Zygmund, Hardy and Littlewood, F. Riesz, and M. Stein.

František Wolf (Berkeley, Calif.).

Bourion, Georges. L'indicatrice de croissance d'une fonction sous-harmonique de n variables. Ann. Sci. École Norm. Sup. (3) **65**, 1–10 (1948).

The author gives an extension of his earlier study of the growth indicatrix [Bull. Sci. Math. (2) **71**, 17–25 (1947); these Rev. **9**, 352] to subharmonic functions of n variables.

M. Heins (Providence, R. I.).

Fichera, Gaetano. Teoremi di completezza connessi all'integrazione dell'equazione $\Delta_4 u=f$. Giorn. Mat. Battaglini (4) **77**, 184–199 (1947).

L'auteur étudie en se plaçant pour le langage dans l'espace ordinaire les solutions de $\Delta_4 u=0$ dans un domaine fini dont la frontière est formée d'un nombre fini de surfaces à courbures continues. Il montre essentiellement la propriété de "complétion hilbertienne" sur la frontière, de certains vecteurs à deux composantes $(v_i, \partial v_i/\partial n)$, où les v_i sont, dans le cas d'une frontière connexe, une suite de polynomes homogènes harmoniques et bihyperharmoniques permettant d'obtenir tous ceux-ci par combinaison linéaire finie; cette complétion signifie que si (α, β) est un vecteur fonction d'un point-frontière, les conditions $\int(\alpha v_i+\beta\partial v_i/\partial n)d\sigma=0$ ($d\sigma$ mesure superficielle) entrainent $\alpha=\beta=0$. L'auteur s'appuie sur des propriétés du potentiel de simple et double couche et notions analogues (obtenues en remplaçant $1/r$ par r) lorsqu'il y a une densité superficielle sommable, propriétés qui généralisent des résultats classiques de continuité et discontinuité et que sont annoncées comme démontrées dans un autre travail [à paraître]. Mais je dois signaler que ces résultats préliminaires concernant le potentiel classique sont en grande partie contenus dans des études approfondies relatives à des distributions de masses quelconques sur des surfaces même un peu moins régulières [G. C. Evans et E. R. C. Miles, Amer. J. Math. **53**, 493–516 (1931); G. C. Evans, ibid. **54**, 213–234 (1932); **55**, 29–49 (1933)].

M. Brelot (Grenoble).

Differential Equations

Saltykow, N. Méthode de d'Alembert pour intégrer les équations différentielles ordinaires linéaires à coefficients constants. Acad. Serbe Sci. Publ. Inst. Math. **2**, 190–204 (1948). (French. Serbian summary)

The method of integrating factors is applied to find integrals of a system of equations whose characteristic equation has multiple roots.

P. Franklin (Cambridge, Mass.).

Fraeys de Veubeke, B. Déphasages caractéristiques et vibrations forcées d'un système amorti. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 626–641 (1948).

The author discusses the response of a linear dissipative system to forced oscillations of a given frequency by using a generalization of the notion of normal modes for a conservative system.

P. Franklin (Cambridge, Mass.).

Cafiero, Federico. Su due teoremi di confronto relativi ad un' equazione differenziale ordinaria del primo ordine. Boll. Un. Mat. Ital. (3) **3**, 124–128 (1948).

The author gives another proof of a theorem of Bajada. Let $f(x, y)$ be measurable with respect to x, and continuous with respect to y in the strip $x_0 \leqq x \leqq x_0+a$, $-\infty<y<\infty$, where $a>0$, and satisfy the condition $|f(x, y)| \leqq q(x)$, where $q(x)$ is Lebesgue integrable over (x_0, x_0+a). Then if $g(x)$ is continuous and satisfies the integral inequality

$$g(x) \leqq g(z)+\int_z^x f(t, g(t))dt, \quad x_0 \leqq z \leqq x \leqq x_0+a,$$

there will exist at least one solution $G(x)$ of the integral equation

$$y=y_0+\int_{x_0}^x f(t, y(t))dt, \qquad y_0 \geqq g(x_0),$$

satisfying the inequality $G(x) \geqq g(x)$ in (x_0, x_0+a).

R. Bellman (Stanford University, Calif.).

Wallach, Sylvan. The differential equation $y'=f(y)$. Amer. J. Math. **70**, 345–350 (1948).

This paper extends known results on the existence of solutions of $y'=f(y)$ [cf. E. Kamke, Differentialgleichungen reeller Funktionen, Leipzig, 1930, pp. 7–21]. It is shown that, if $f(y)$ is continuous for $c \leqq y \leqq d$, then the equation has a nonconstant solution in an interval $0 \leqq x \leqq b$, satisfying the initial condition $y(0)=c$, if and only if $f(y) \geqq 0$ for some interval $c \leqq y \leqq k$ and $\int_c^k \{f(s)\}^{-1}ds$ exists. The nonconstant solutions are then analyzed in detail.

W. Kaplan.

Vasil'eva, A. B. On the differentiation of solutions of differential equations containing a small parameter. Doklady Akad. Nauk SSSR (N.S.) **61**, 597–599 (1948). (Russian)

The author considers the system of differential equations $\dot x=f(t, x, z)$, $\epsilon\dot z=F(t, x, z)$, where x is a vector with n components as is f while z is a scalar. Under certain hypotheses of Tihonov the solutions for small $\epsilon>0$ with fixed initial conditions tend to a continuous solution of the degenerate system in which $\epsilon=0$. The author considers the limiting values of the derivatives of these solutions, $(\dot x(t, \epsilon), \dot z(t, \epsilon))$, $(\ddot x(t, \epsilon), \ddot z(t, \epsilon))$, etc. as $\epsilon\to 0$ and states hypotheses under which these converge to the corresponding derivatives of the solution of the degenerate system. The derivative with respect to ϵ is also considered. [Actually the results of Friedrichs and Wasow [Duke Math. J. **13**, 367–381 (1946); these Rev. **8**, 272] are considerably more general in that they consider derivatives with respect to initial conditions. By a standard artifice this includes the derivatives with respect to t as a special case. The treatment for the first derivative goes over to higher derivatives at once and the differentiation with respect to ϵ also follows exactly the same argument. In fact results for derivatives with respect to initial conditions have also been stated for cases where the degenerate system has a discontinuous solution [Levinson, Proc. Nat. Acad. Sci. U. S. A. **33**, 214–218 (1947); these Rev. **9**, 144].]

N. Levinson (Copenhagen).

Cartwright, Mary L. Forced oscillations in nearly sinusoidal systems. J. Inst. Elec. Engrs. Part III. **95**, 88–96 (1948).

In this paper the differential equation

$$\ddot v-(\alpha+\beta v-\gamma v^2)\dot v+\omega^2 v=E\omega_1^2\sin\omega_1 t$$

is analyzed in great detail for certain ranges of the parameters α, β, γ, ω, ω_1. It is assumed in particular that α, β, γ and $\omega-\omega_1$ are small compared to ω. The results are presented without proof. They consist of a classification of various types of periodic or quasi-periodic solutions. Numerous diagrams give a full qualitative and quantitative picture of the different cases. *W. Kaplan* (Ann Arbor, Mich.).

Sestini, Giorgio. Moto di un punto soggetto a resistenza e a forza di richiamo. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **10(79)**, 117–134 (1946).

The author considers the equation

$$m\ddot{x}+k|\dot{x}|\dot{x}+2\epsilon x+\omega^2x=0,$$

where all the constants are positive. He shows that if $\epsilon\geqq\omega$ all integrals are nonoscillatory, while if $\epsilon<\omega$ all integrals are oscillatory. The asymptotic behavior of $x(t)$ and $\dot{x}(t)$ is discussed in the oscillatory case. [It may be noted that the asymptotic behavior of nonoscillatory solutions can be determined quite precisely using a result due to Hardy; cf. Fowler, Quart. J. Math., Oxford Ser. 2, 259–288 (1931), p. 270; or Hardy, Proc. London Math. Soc. (2) 10, 451–468 (1911).] *R. Bellman* (Stanford University, Calif.).

Armellini, Giuseppe. Sopra una classe di equazioni differenziali della meccanica celeste di cui l'integrale generale tende a zero. I. Pont. Acad. Sci. Acta **6**, 387–396 (1942).

A study is made of the asymptotic properties of the solutions of the differential equation $\ddot{r}=c^2r^{-3}-f(r)M(t)$. It is assumed that f is positive for positive r, is bounded near $r=0$, and does not approach 0 as r becomes infinite, and that M is positive and nondecreasing for positive t. It is shown that, if M is bounded, then, as t becomes infinite, a solution $r(t)$ must either approach a positive limit r_1 or oscillate between successive maxima and minima which are themselves respectively decreasing. If M is unbounded, a solution $r(t)$ must either approach 0 as t becomes infinite or else oscillate, the successive minima approaching 0.

W. Kaplan (Ann Arbor, Mich.).

Rocard, Yves. Sur les conditions d'auto-oscillation des systèmes vibrants. Proc. Phys. Soc. **61**, 393–402 (1948). Expository lecture.

Volk, I. M. On the stability of periodic motions when the equations and their periodic solutions are known only approximately. Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 647–650 (1948). (Russian)

Consider a system of ordinary first order differential equations whose right members are meromorphic functions of a small parameter μ. If all but the leading terms with respect to μ are neglected, a simplified system $dx_\nu/dt=\mu^{k_\nu}X_\nu(x_1,\cdots,x_n;t)$ $(\nu=1,2,\cdots,n)$ is obtained in which the k_ν may be positive or negative integers. In three previous papers [cf. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 559–574 (1946); same journal **11**, 433–444 (1947); **12**, 29–38 (1948); these Rev. **8**, 330; **9**, 185, 538] the author investigated the existence, for small μ, of periodic solutions x_ν^* of the full system near a given periodic solution x_ν^0 of the simplified system. In the present paper it is shown that for sufficiently small $|\mu|$ the stability character in the sense of Liapounoff is the same for the x_ν^* as for the x_ν^0. *W. Wasow* (Swarthmore, Pa.).

Yurovskiĭ, A. V. On certain criteria for the stability of the integrals of a system of two linear differential equations with periodic coefficients. Doklady Akad. Nauk SSSR (N.S.) **62**, 595–598 (1948). (Russian)

The author proves the following generalization of a classical theorem of Liapounoff. Consider the system

$$(1)\qquad dx_i/dt=p_{i1}(t)x_1+p_{i2}(t)x_2,\qquad i=1,2,$$

where $p_{ij}(t)$ is real, continuous, and of period $\omega>0$. If $\int_0^\omega p_{11}(t)dt=\int_0^\omega p_{22}(t)dt=\alpha\leqq0$, p_{12} and p_{21} are neither identically zero, do not change sign over a period, and are of different sign, and

$$(2)\quad \int_0^\omega|p_{12}|\exp\left\{\int_0^t(p_{22}-p_{11})dt_1\right\}dt \times\int_0^\omega|p_{21}|\exp\left\{\int_0^t(p_{11}-p_{22})dt_1\right\}dt\leqq4,$$

then all solutions are bounded as $t\to\pm\infty$. A similar, more complicated, result is given for the case where p_{12} and p_{21} are of the same sign. *R. Bellman.*

Persidskiĭ, K. P. On the stability of the solution of an infinite system of equations. Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 597–612 (1948). (Russian)

The author considers infinite systems of differential equations of the type (1) $dx_s/dt=w_s(x_1,x_2,\cdots,t)$, $s=1,2,\cdots$, where it is assumed, inter alia, that

$$|w(x_1,x_2,\cdots,t)-w(x_1',x_2',\cdots,t)|\leqq A(t)|\sum_{i=1}^\infty a_{si}|x_i-x_i'||,$$

$s=1,2,\cdots$, in the region defined by $t\geqq0$, $|x_i|\leqq R>0$, $\sum_{i=1}^\infty a_{si}\leqq L$, and that $w_s(0,0,\cdots,t)=0$, $s=1,2,\cdots$. In the first part of the paper, using the method of successive approximations, the existence of a solution determined by the initial conditions $x_s(0)=c_s$ is demonstrated. It is also shown that a "méthode des réduites" is valid, namely, the solution of $dx_s/dt=w_s(x_1,x_2,\cdots,x_N,0,0,\cdots,t)$ approaches the solution of (1) as $N\to\infty$.

In the second part of the paper, the author turns to the question of the stability of the "trivial" solution, $x_s=0$, $s\geqq1$. First the homogeneous equation of first approximation is discussed, $dx_s/dt=\sum_{i=1}^\infty p_{si}(t)x_i(t)$, $s=1,2,\cdots$; then the nonhomogeneous linear equation, $dx_s/dt=\sum_{i=1}^\infty p_{si}(t)x_i(t)+f_s(t)$, and then finally, using the second method of Liapounoff, the nonlinear equation (1), all under certain hypotheses on w_s, $p_{si}(t)$ and $f_s(t)$. *R. Bellman* (Stanford University, Calif.).

Persidskiĭ, K. On the characteristic numbers of the solution of an infinite system of linear differential equations. Doklady Akad. Nauk SSSR (N.S.) **63**, 229–232 (1948). (Russian)

The author applies the concept of a characteristic number of a solution of a linear differential equation, introduced by Liapounoff, to infinite systems of linear differential equations of the form $dx_s/dt=\sum_{i=1}^\infty p_{si}(t)x_i$, $s=1,2,\cdots$. Thus, for example, he proves that if $\sum_{i=1}^\infty|p_{si}(t)|\leqq p(t)$, $s\geqq1$, and if one sets $x(t)=\sup_i|x_i(t)|$, then $-\int_{t_0}^t p(t_1)dt_1\leqq\log x(t)/x(t_0)\leqq\int_{t_0}^t p(t_1)dt_1$.

R. Bellman (Stanford University, Calif.).

Povzner, A. On differential equations of Sturm-Liouville type on a half-axis. Mat. Sbornik N.S. **23(65)**, 3–52 (1948). (Russian)

The author studies the operator $L(u)=d^2u/dx^2-\rho(x)u$ $(\rho(-x)=\rho(x))$ on the semiaxis. When (1) $\partial^2u/\partial x^2-\rho(x)u$

$=\partial^2u/\partial y^2-\rho(y)u$, with boundary conditions $u(x,0)=f(x)$, $u_y(x,0)=0$, then u is an additive operator $T_x^y(f)$. If $\varphi(x,\lambda)$ satisfies $L(y)+\lambda(\varphi)=0$, $\varphi(0,\lambda)=1$, $\varphi'(0,\lambda)=0$, then (2) $T_x^y(\varphi(x,\lambda))=\varphi(x,\lambda)\varphi(y,\lambda)$, which is a generalization of a formula for $\cos\lambda x$ (obtained for $\rho=0$). A "product" is defined by (3) $f\circ g=\int_0^\infty T_x^y(f)g(y)dy$. When f is even, it is shown that (4) $T_x^y(f)=\frac{1}{2}[f(x+y)+f(x-y)]-\int_0^\infty f(t)w(t,x,y)dt$. If $\rho(x)=O(x^{-3-\epsilon})$, then $\omega=u_1'(0)w_1(t,x,y)+w_2(t,x,y)$; here $\int_0^\infty|w_2|dt\leqq K$ and w_1 is defined by $\int_{x-y}^{x+y}f(t)dt=\int_0^\infty w_1dt$; if $\rho(x)=O(x^{-3-\epsilon})$, $|w|$ is bounded. The above allows constructions of a suitable normed ring and certain evaluations for the characteristic functions $\varphi(x,\lambda)$ for λ small. Operators P, Q are studied such that $P(\cos(\lambda^{\frac{1}{2}}x))=\varphi(x,\lambda)$, $Q(\varphi(x,\lambda))=\cos(\lambda^{\frac{1}{2}}x)$. If $\rho(x)=O(x^{-3-\epsilon})$, then $|\varphi(x,\lambda)|<K$ (all $\lambda\geqq 0$) if and only if $|\varphi(x,0)|<K$; if $\rho(x)=O(x^{-2-\epsilon})$, then $|\varphi(x,\lambda)|\leqq K_1(|x|+1)$ $(\lambda\geqq 0)$; if $\varphi(x,\lambda)=O(|x|)$, then λ is real. Suppose $\rho(x)=O(x^{-2-\epsilon})$; in order that the set of functions f with norm $\|f\|=\int_0^\infty|f|dt$ be a normed commutative ring, with multiplication according to (3), it is necessary and sufficient that $\varphi(x,0)$ be bounded; in the latter case, the set of all bounded $\varphi(x,\lambda)$ defines the maximal ideals of this ring. If $\rho(x)=O(x^{-2-\epsilon})$, then the set of functions f with norm $\|f\|=\int_0^\infty(1+t)|f(t)|dt$ is a normed commutative ring (multiplication as in (3)); the maximal ideals of this ring are defined by the $\varphi(x,\lambda)$ such that $|\varphi|=O(x)$. The results of this work enable a generalization of a theorem of Bochner about positive functions and of a theorem of Plancherel. The author intends to study theorems of Plancherel type in a later work. *W. J. Trjitzinsky.*

Sansone, Giovanni. Studi sulle equazioni differenziali lineari omogenee di terzo ordine nel campo reale. Univ. Nac. Tucumán. Revista A. **6**, 195–253 (1948).

This paper studies an ordinary linear homogeneous differential equation of the third order with real independent variable. For the most part, Birkhoff's canonical form (1) $y'''+2A(x)y'+[A'(x)+\omega(x)]y=0$, where A, A', ω are real and continuous, is the point of departure. Among the new results are the following: (i) an oscillation theorem for an equation $y'''+2A(\lambda)y'+\Omega(\lambda)y=0$, where λ denotes a real parameter, and an existence theorem for the characteristic values of a corresponding system; (ii) for comparing $y'''+py'+qy=0$ with $y'''+Py'+Qy=0$, where $p\leqq P$, $0\leqq q$, $0<Q$, a theorem based on an identity like Picone's for the second order equation; (iii) a generalization of a theorem of Birkhoff [Ann. of Math. (2) 12, 103–127 (1911), pp. 121–122] for comparing (1) with the self-adjoint equation got by making $\omega=0$ in (1); (iv) construction of an equation of form (1) every solution of which has not fewer than an arbitrarily prescribed number of zeros on a given interval; (v) conversion of the system $y'''+2\lambda A(x)y'+\lambda[A'(x)+\omega(x)]y=0$, $y(a)=y'(a)=y(b)=0$, into a Fredholm integral equation and construction of particular examples in which there are respectively at least one, infinitely many and no characteristic values; and (vi) a discussion of point pairs conjugate with respect to (1). In addition, there is a useful survey of results obtained principally by Birkhoff, Mammana and Ascoli in relation to the aspects of the subject covered by this paper. *J. M. Thomas* (Durham, N. C.).

Kasner, Edward, and De Cicco, John. Osculating conics of the integral curves of third order differential equations of the type (G). Proc. Nat. Acad. Sci. U. S. A. **35**, 43–46 (1949).

This note gives some new results concerning the properties of families of curves defined by differential equations of the form $y'''=G(x,y,y')y''+H(x,y,y')y''^2$. Such families of curves arise in many geometrical and physical problems. The principal new results can be stated briefly as follows. If, for each of the ∞^1 curves which pass through a fixed point P in a fixed direction, we construct the conic which osculates the curve at P, these conics are all tangent to another conic in general position. Also, the centers of the osculating conics describe still another conic passing through P; and the foci of the osculating conics describe an algebraic curve of the sixth degree with a singular point of the fourth order at P. *L. A. MacColl* (New York, N. Y.).

Ostrowski, Alexandre. Sur le rayon de convergence de la série de Blasius. C. R. Acad. Sci. Paris **227**, 580–582 (1948).

The author considers the radius of convergence of the power series solution of $y'''(x)+yy''(x)=0$, $y(0)=y'(0)=0$, $y''(0)=1$. Ovdart had shown that the radius R satisfies $2.884<R<3.203$. Using two methods, one based on the elementary demonstration of Borel of the Picard theorem and the second based on majorant series the author proves $3.1<R<3.18$. He can also sharpen these results.
N. Levinson (Copenhagen).

Fichera, Gaetano. Sui differenziali totali di qualsivoglia ordine. Boll. Un. Mat. Ital. (3) **3**, 105–108 (1948).

For fixed but arbitrary n let the nth derivatives of an unknown F be put equal to given functions which involve the r independent variables and which have first derivatives satisfying the integrability conditions. This note states in terms of integrals evaluated along a certain system of closed curves necessary and sufficient conditions for the existence of F. It generalizes to r dimensions the result stated for two dimensions by M. Picone [Boll. Un. Mat. Ital. (1) 16, 77–82 (1937)]. *J. M. Thomas* (Durham, N. C.).

Cramlet, Clyde M., Muggli, Ethel C., and Zuckerman, Herbert S. On systems of partial differential equations. Univ. Washington Publ. Math. **3**, no. 1, 45–54 (1948).

The terminology and notation of the theory of classes are employed to discuss the sets of derivatives known as the principal and parametric sets in the Riquier theory of systems of partial differential equations. The results are applied to the system which specifies a finite set of derivatives of a single unknown as functions of the independent variables alone. Such a system can be regarded as a generalization of the system which specifies all the first derivatives. For a passive system of this kind the solution is given a new form which involves an integration operator and which is a generalization of the ordinary formula for integrating an exact differential. *J. M. Thomas* (Durham, N. C.).

Bicadze, A. V. On the uniqueness of the solution of the Dirichlet problem for elliptic partial differential equations. Uspehi Matem. Nauk (N.S.) **3**, no. 6(28), 211–212 (1948). (Russian)

Let A, B, C be given square matrices (depending on x and y), u an unknown vector (depending on x and y). The equation (1) $Au_{xx}+2Bu_{xy}+Cu_{yy}=0$ is called elliptic if the equation $\det|A\alpha^2+2B\alpha+C|=0$ (α a scalar) has no real roots. The author gives two examples of equations (1) (with constant coefficients A, B, C) which possess solutions $u\neq 0$ vanishing for $x^2+y^2=R^2$ and regular for $x^2+y^2<R^2$.
L. Bers (Syracuse, N. Y.).

Pleijel, Åke. Asymptotic relations for the eigenfunctions of certain boundary problems of polar type. Amer. J. Math. **70**, 892–907 (1948).

Let V be a bounded connected and open domain in Euclidean 3-space and S its boundary. The author considers the eigenvalue problem (*) $\{\Delta-(p-\lambda q)\}u=0$ and $u=0$, or $\partial u/\partial n=0$, on S. The problem is of polar type; i.e., q changes sign in V. Following a method due to Carleman [C. R. Huitième Congrès Math. Scandinaves 1934, pp. 34–44 (1935)] the author obtains the following asymptotic relation, valid when $q(P)\neq 0$, for the eigenfunctions:

$$(^{**})\qquad \sum_{0<\lambda_n<T}\phi_n^2(P)=[q_+^{\frac{3}{2}}(P)/6\pi^2]T^{\frac{3}{2}}+o(T^{\frac{3}{2}}),$$

where $q_+(P)$ is the "positive part" of q, and a similar relation for the sum from $-T$ to 0, involving the "negative part" of q. Denoting by $G(P,Q;\lambda)$ the Green's function of the eigenvalue problem, the validity of the development

$$(^{***})\qquad \lim_{Q\to P}\{G(P,Q;\lambda)-G(P,Q;0)\}=\sum_{n=-\infty}^{\infty}\phi_n^2(P)/\{\lambda_n(\lambda_n-\lambda)\}$$

is verified for complex λ, when $q(P)\neq 0$. A consideration of the asymptotic behavior of (***) as $\lambda\to\infty$ through imaginary values enables the author to employ a Tauberian theorem to obtain (**). *M. J. Gottlieb* (Newark, N. J.).

Samarsky, A. Concerning a problem of the transfer of heat. Vestnik Moskov. Univ. **1947**, no. 3, 85–101 (1947). (Russian. English summary)

A finite cylindrical bar is placed in contact with a heat reservoir at the end $x=0$, with Newton's law of heat transfer applying at $x=0$. Heat is gained (or lost) by the bar at the end $x=l$ and through the lateral surface (though radial flow within the bar is neglected), and heat may also be gained by the reservoir, Newton's law again applying to the flow of heat. Temperatures of the reservoir and of sections of the bar are found by a method similar to that of R. E. Langer [Tôhoku Math. J. **35**, 260–275 (1932)]. A section is devoted to the use of this problem in determining the physical constants involved. *R. E. Gaskell* (Ames, Iowa).

Samarskiĭ, A. A. Concerning a problem of the transfer of heat. II. Vestnik Moskov. Univ. **1947**, no. 6, 119–129 (1947). (Russian)

Loaded integral equations are used to establish the solution given in the paper reviewed above. A uniqueness proof is also given. *R. E. Gaskell* (Ames, Iowa).

Benfield, A. E. A problem of the temperature distribution in a moving medium. Quart. Appl. Math. **6**, 439–443 (1949).

A semi-infinite medium $x\geqq 0$ with initial temperatures proportional to x moves with a negative constant velocity $-v$ past the plane $x=0$. Whenever a section of the medium coincides with the plane $x=0$ the temperature of that section is reduced to zero. In terms of the coordinate x in the fixed reference system then the temperature function $T(x,t)$ satisfies the boundary value problem $T_t=kT_{xx}+vT_x$, $T(x,0)=\lambda x$, $T(0,t)=0$. With the aid of elementary properties of the Laplace transformation a formula is derived for $T(x,t)$ in terms of error functions. The author refers to a forthcoming paper for a geophysical interpretation of the problem. *R. V. Churchill* (Ann Arbor, Mich.).

Bureau, Florent. Sur l'intégration des équations linéaires aux dérivées partielles simplement hyperboliques, par la méthode des singularités. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 480–499 (1948).

In the theory of the wave equation

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x_1^2}+\cdots+\frac{\partial^2 u}{\partial x_n^2}$$

(and similarly in the study of the elementary solution for more general hyperbolic equations of second order) there appears a striking difference in behavior for even n and for odd n. J. Hadamard introduced the notion of "finite part" of a divergent integral to obtain a simple expression for the solution in the case of even n. He then reduced the case of odd n to that of even n by the "method of descent." The solution for odd n can also be obtained directly by introducing the "logarithmic part" of a divergent integral [see Courant and Hilbert, Methoden der mathematischen Physik, v. 2, Springer, Berlin, 1937, p. 437]. Though other methods, like that of M. Riesz and Mathisson, will yield the solution of the wave equation, the author believes that the use of finite and logarithmic parts is essentially unavoidable in the study of the general Cauchy problem. In the present note the author gives a simple exposition of this "method of singularities." He shows in detail how it comes about that the method of descent, starting with the expression of the solution for even n as finite part of a divergent integral, will yield a representation of the solution for odd n in the form of the logarithmic part of another divergent integral. *F. John* (New York, N. Y.).

Caldirola, Piero, e Sillano, Pietro. Integrazione dell'equazione delle onde sferiche smorzate e forzate col calcolo simbolico. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **10(79)**, 55–68 (1946).

The equation $U_{tt}-\nabla^2U-kU=X$, with general conditions on the boundary and at $t=0$, is solved formally with the help of the Laplace transformation. *R. E. Gaskell.*

Petrovskiĭ, I. G. On some problems of the theory of partial differential equations. Uspehi Matem. Nauk (N.S.) **1**, no. 3-4(13-14), 44–70 (1946). (Russian)

This is a survey of typical questions arising in the theory of partial differential equations, with many indications as to problems awaiting solution. Briefly noted, the content of the three parts into which the survey is divided is as follows. (I) Hyperbolic equations and Cauchy's problem. S. Kowalevsky's theorem [J. Reine Angew. Math. **80**, 1–32 (1875)] for the solution of Cauchy's problem for the equation

$$\partial^n u/\partial t^n=F\left(t,x_1,\cdots,x_p,u,\cdots,\frac{\partial^n u}{\partial t^{k_0}\partial x_1^{k_1}\cdots\partial x_p^{k_p}}\right),$$

$k_0<n$, the given analytic data being $\partial^k u/\partial t^k=\varphi_k(x_1,\cdots,x_p)$, $k=0,1,\cdots,n-1$ on the plane $t=0$; Holmgren's [Ofversigt af Kongl. Svenska Vetenskaps-Akad. Förhandlingar **58**, 91–103 (1901)] and Carleman's [Ark. Mat. Astr. Fys. **26B**, no. 17 (1939); these Rev. **1**, 55] uniqueness theorems for the related problem where the function is sought in a domain on one side of, and bounded partially by, the plane $t=0$; Hadamard's example of an "incorrectly posed" boundary value problem [Le Problème de Cauchy, Hermann, Paris, 1932, pp. 40–41]; following Hadamard's ideas, a definition is given of a correctly posed boundary value problem, and certain sufficient conditions for this to be the case are dis-

cussed [for hyperbolic equations Cauchy's problem is always correctly posed: see the author's paper, Rec. Math. [Mat. Sbornik] N.S. **2**(44), 815–870 (1937)]; Hadamard's result [Le Problème de Cauchy, Hermann, Paris, 1932, pp. 239–241] concerning linear hyperbolic second order equations in an odd number of variables, for which "diffusion" always occurs; M. Mathisson's results [Acta Math. **71**, 249–282 (1939); these Rev. **1**, 120] when the number of variables is four, and the author's discussion of the general hyperbolic equation in this connection [C. R. (Doklady) Acad. Sci. URSS (N.S.) **38**, 151–153 (1943); these Rev. **5**, 8].

(II) Elliptic equations. Dirichlet problem for Laplace's equation; method of subharmonic functions, regular boundary points [O. Perron, Math. Z. **18**, 42–54 (1923); N. Wiener, J. Math. Physics **3**, 129–146 (1924)]; Lebesgue's [C. R. Soc. Math. France **1913**, 17] and Urysohn's [Math. Z. **23**, 155–158 (1925)] examples; Lavrentieff and Keldych's results [Keldych, Uspehi Matem. Nauk **8**, 171–231 (1941); these Rev. **3**, 123] on the "stability" of the solution of the Dirichlet problem when the given domain G is "approached" by means of domains contained in, and containing, it; method of finite differences; the author [Uspehi Matem. Nauk **8**, 107–114, 161–170 (1941); these Rev. **3**, 123] proves that the function so obtained satisfies the boundary conditions; Neumann's problem; boundary value problems for more general elliptic equations and systems [Feller, Math. Ann. **102**, 633–649 (1930); Z. Shapiro, C. R. (Doklady) Acad. Sci. URSS (N.S.) **46**, 133–135 (1945); N. I. Simonov, ibid. **44**, 259–261 (1944); these Rev. **7**, 14; **6**, 228]; Lavrentieff's [Rec. Math. [Mat. Sbornik] **42**, 407–424 (1935)] quasi-conformal maps, defined by functions u and v satisfying a linear elliptic system $\partial u/\partial x=a_1\partial u/\partial y+b_1\partial v/\partial y$, $\partial v/\partial x=a_2\partial u/\partial y+b_2\partial v/\partial y$, and for which, in particular, the generalization of Riemann's conformal mapping theorem has been obtained; Σ-monogenic functions, introduced by Bers and Gelbart [Trans. Amer. Math. Soc. **56**, 67–93 (1944); these Rev. **6**, 86] are discussed at length, and related unpublished work of Markuševič and others is mentioned.

(III) Parabolic equations. This section deals mostly with the results of Tichonoff [Bull. Math. Univ. Moscou A **1**, no. 9 (1938)]. At the end, L. Ásgeirsson's [Math. Ann. **113**, 321–346 (1936)] mean value theorem for ultrahyperbolic equations is discussed. *J. B. Diaz* (Providence, R. I.).

Myškis, A. The uniqueness of the solution of Cauchy's problem. Uspehi Matem. Nauk (N.S.) **3**, no. 2(24), 3–46 (1948). (Russian)

Given a system of differential equations

$$F_j(x_1, \cdots, x_n, z_1, \cdots, z_m, \partial z_1/\partial x_1, \cdots)=0, \qquad j=1, \cdots, k,$$

let G be a domain in $(x_1, \cdots, x_n)$ space, Γ the boundary of G, and M a subset of Γ. The "local" Cauchy problem consists in finding (in a certain class of functions) a solution $z_1, \cdots, z_m$ of (1), defined on all points of G "sufficiently near" to M, and such that for each $i\,(=1, \cdots, m)$, the function z_i and its partial derivatives up to order l_i inclusive assume given values (called the "initial data") on M. The uniqueness problem (for the given class of functions) consists in finding conditions under which two solutions of (1), satisfying the same initial data, must coincide for all points of G sufficiently near to M. The present paper is a review of the literature on the subject, and contains many proofs and several improvements of the results available. A bibliography of sixty articles is included.

Section 1 deals with the period up to 1901, principal mention being made of the treatment of the "normal" case of (1) by S. Kowalevsky [J. Reine Angew. Math. **80**, 1–32 (1875)]. Here

$$\partial^{p_i}z_i/\partial x_1^{p_i}=f_i(x_1, \cdots, x_n, z_1, \cdots, z_m, \partial z_1/\partial x_1, \cdots), \qquad i=1, \cdots, m,$$

where the functions f_i do not contain the arguments $\partial^q z_j/\partial^r x_1 \cdots$ for $q>p_j$ or $r\geqq p_j$. Also, G is the half-space $x_1>0$, Γ is the plane $x_1=0$, and M is open in Γ; further, $l_i=p_i-1$ $(i=1, \cdots, m)$. In this connection, reference is made to the recent work of N. A. Lednev [Mat. Sbornik N.S. **22**(64), 205–266 (1948); these Rev. **10**, 253]. Section 2 contains a proof of Holmgren's [Ofversigt af Kongl. Svenska Vetenskaps-Akad. Förhandlingar **58**, 91–103 (1901)] uniqueness theorem for linear systems of arbitrary order in any number of independent variables and with analytic coefficients, the solutions of the system being assumed to be continuous together with all the derivatives which appear in the system; and mentions Hadamard's [Leçons sur la Propagation des Ondes et les Équations de l'Hydrodynamique, Paris, 1903, note 1] remark that the uniqueness problem for nonlinear systems rests on the uniqueness problem for linear systems, whose coefficients, however, need not be analytic even if the functions involved in the given nonlinear system are. Section 3 takes up the case of a single equation of the first order, which occupies a special place in view of its relation to certain systems of ordinary differential equations. Haar's [Atti Congresso Internaz. Mat., Bologna, 1928, v. 3, pp. 5–10 (1930)] lemma is proved and various of its consequences are presented. Section 4 treats hyperbolic equations and systems, again using Haar's lemma and Hadamard's remark. Section 5 is concerned with elliptic equations and systems. In particular, T. Carleman's [C. R. Acad. Sci. Paris **197**, 471–474 (1933)] results concerning the system

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=\alpha(x, y)u+\beta(x, y)v;$$

$$\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=\gamma(x, y)u+\delta(x, y)v,$$

are considered in detail. Section 6 discusses Carleman's [Ark. Mat. Astr. Fys. **26B**, no. 17 (1939); these Rev. **1**, 55] results for the normal system

$$\frac{\partial z_p}{\partial x}+\sum_{q=1}^{n}A_{pq}(x, y)\frac{\partial z_q}{\partial y}+\sum_{q=1}^{n}B_{pq}(x, y)z_q=0, \qquad p=1, \cdots, n.$$

Section 7 contains an example given by the author [Doklady Akad. Nauk SSSR (N.S.) **58**, 21–24 (1947); these Rev. **9**, 354] of the nonuniqueness of the solution of Cauchy's problem for

$$\frac{\partial u}{\partial x}=a_1(x, y)\frac{\partial u}{\partial y}+b_1(x, y)\frac{\partial v}{\partial y},$$

$$\frac{\partial v}{\partial x}=a_2(x, y)\frac{\partial u}{\partial y}+b_2(x, y)\frac{\partial v}{\partial y},$$

which shows that Haar's method used in section 3 cannot be directly carried over to deal with such a system. Section 8 considers the uniqueness of the "nonlocal" Cauchy problem, where the solutions are required to be defined on all of G. Reference is made to the author [Rec. Math. [Mat. Sbornik] N.S. **19**(61), 489–522 (1946); these Rev. **8**, 383], to F. John [Proc. Nat. Acad. Sci. U. S. A. **29**, 98–104

(1943); these Rev. 4, 279], and to an example of T. Ważewski of an infinitely differentiable function $Q(x, y)$ defined on a simply connected domain G and such that any continuous solution in G of the equation $\partial u/\partial x+Q(x, y)\partial u/\partial y=0$ is a constant. *J. B. Diaz* (Providence, R. I.).

Robinson, A. On the integration of hyperbolic differential equations. Coll. Aeronaut. Cranfield. Rep. no. 18, 26 pp. (1948).

The author proves the existence and uniqueness of the solution of a linear hyperbolic equation of the form

$$(1) \qquad \sum_{k=0}^{n}\sum_{m=0}^{k} a_{km}(x, y)\frac{\partial^k z}{\partial x^{k-m}\partial y^m}=a_0(x, y),$$

assuming prescribed Cauchy data on $x=0$. This problem has been solved previously, even for the nonlinear case [see, e.g., M. Cinquini-Cibrario, Ann. Mat. Pura Appl. (4) 24, 157–175 (1945); these Rev. 9, 441]. However, the linear case permits considerable simplifications. The author shows that (1) can be reduced to the solution of the Cauchy problem for a system of first order linear equations of the form

$$\frac{\partial f_i}{\partial x}+\gamma_i(x, y)\frac{\partial f_i}{\partial y}=\sum_{k=1}^{n} b_{ik}f_k+a_i, \quad i=1, \cdots, n,$$

where the γ_i are the characteristic roots of (1) and the f_i are "suitable" linear combinations of z and its derivatives. However, in order to determine the coefficients of those combinations one solution of a more general nonlinear system of the form

$$(2) \qquad \frac{\partial f_{ik}}{\partial x}+\gamma_i(x, y)\frac{\partial f_{ik}}{\partial y}=F_{ik}(x, y, f_{11}, \cdots, f_{nm})$$

$(i=1, \cdots, n; k=1, \cdots, m)$ has to be obtained, where the F_{ik} are quadratic in the f_{ik}. The system (2) is reduced to a system of integral equations along the characteristic curves, which can be solved by successive approximations of the type of Picard, and whose solution in turn can be shown to yield a solution of the original problem, if suitable differentiability and Lipschitz conditions are assumed for the coefficients. *F. John* (New York, N. Y.).

Difference Equations, Special Functional Equations

Strodt, Walter. Linear difference equations and exponential polynomials. Trans. Amer. Math. Soc. 64, 439–466 (1948).

The class of equations considered has the form (1) $\sum_{j=1}^{n} A_j y(x+\omega_j)=\phi(x)$, where all quantities are complex, and A_j, ω_j are constants, the ω_j being distinct. Function $\phi(x)$ is of type (M, β); i.e., $\phi(x)$ is analytic in the sector $S(\beta)$: $|\arg x|<\beta\leqq\pi/2$, and to every positive ϵ, δ corresponds $C_0(\epsilon, \delta)$ such that $|\phi(x)|<C_0(\epsilon, \delta)e^{|x|(M+\epsilon)}$ throughout $|\arg x|<\beta-\delta$. It is also assumed that $\omega_1=0$, and that ω_j $(j>1)$ lie in $S(\beta)$. It is shown that equation (1) has a solution $y_0(x)$ of type (M, β), and that the most general solution of this type is given by

$$y(x)=y_0(x)+\sum_{s=1}^{p}\sum_{j=0}^{j_s-1} C_{s,j}x^j e^{\zeta_s x},$$

where the $C_{s,j}$ are arbitrary constants and $\zeta_1, \cdots, \zeta_p$ are the distinct zeros (ζ_s of order j_s) of $f(z)=\sum_{j=1}^{n} A_j e^{\omega_j z}$ in $N(M, \beta)$. This latter region is the set of points x such that, for every β_1 in $-\beta\leqq\beta_1\leqq\beta$, the relation $\Re(x)\cos\beta_1+\Im(x)\sin\beta_1\leqq M$ holds. The proof is made to depend on two features: (i) expressing $1/f(x)$ in a convergent partial-fraction expansion (by suitably grouping the principal parts of $1/f$); (ii) approximating (1) by q-difference equations, solutions of which are found. *I. M. Sheffer* (State College, Pa.).

Ghermanescu, M. Solutions mesurables de certaines équations fonctionnelles linéaires à plusieurs variables. Bull. Sci. Tech. Polytech. Timişoara 13, 18–37 (1948).

The author's objective is to exhibit polynomials or pseudo-polynomials (expressions of the form $\sum x^i A_i(y)+\sum y^j B_j(x)$) as the only solutions, measurable in each variable, of a variety of functional equations. The equations treated in this part of the paper [cf. the following review] are various types of difference equations. At the beginning of the paper a large number of auxiliary theorems are given, such as the following: the general solution, measurable in each variable, of $f(x+y)-f(x)=\varphi(x, y)$, is $f(x)=u(x)+b$, $\varphi(x, y)=u(x+y)-u(x)$, where $u(x)$ is measurable and b is a constant. The various equations are solved by reducing them by changes of variable, etc., to equations whose general solutions were previously known. *R. P. Boas, Jr.*

Ghermanescu, M. Solutions mesurables de certaines équations fonctionnelles linéaires à plusieurs variables. II. Bull. Sci. Tech. Polytech. Timişoara 13, 128–140 (1948).

The author continues part I [reviewed above] by giving the general solution, measurable in each variable, of a number of functional equations. For example, the solutions of $f(x_1+\cdots+x_n, y_1+\cdots+y_n)=\sum_{k=1}^{n} f(\lambda x_k, \mu y_k)$ are of the form $ax+by$ if $\lambda=\mu=1$ and zero otherwise. *R. P. Boas, Jr.*

Stark, M. On a functional equation. Colloquium Math. 1, 230–231 (1948).

The equation is

$$f(\alpha a+(1-\alpha)b)=\frac{k}{b-a}\int_a^b f(t)dt,$$

where α and k are constants and the equation is supposed to hold for all a and b. The author shows that the only solutions are linear. He requires $f(x)$ only to be integrable in some sense, but assumes tacitly that the integral is a continuous function of its upper limit. *R. P. Boas, Jr.*

Aczél, Jean. Sur une équation fonctionnelle. Acad. Serbe Sci. Publ. Inst. Math. 2, 257–262 (1948). (French. Serbian summary)

If the equation (1) $f(ax+by+c)=\Phi[f(x), f(y)]$, in which $\Phi(u, v)$ is continuous and strictly monotonic and $f(z)$ is unknown, has a solution, then Φ necessarily is bisymmetric and therefore Φ is of the form $\Phi(u, v)=\varphi[\alpha\varphi^{-1}(u)+\beta\varphi^{-1}(v)+\gamma]$, where $\varphi(w)$ is continuous and strictly monotonic; then the substitution $F(z)=\varphi^{-1}[f(z)]$ transforms (1) into (2) $F(ax+by+c)=\alpha F(x)+\beta F(y)+\gamma$, which has only the trivial solution $F(z)\equiv\gamma/(1-\alpha-\beta)=$constant, $\alpha+\beta\neq 1$, unless $\alpha=a$, $\beta=b$; in the latter case (2) has the additional solution $F(z)=Az+(\gamma-Ac)/(1-a-b)$, $a+b\neq 1$, where A is an arbitrary constant, whence (1) has the solution $f(z)=\varphi[Az+(\gamma-Ac)/(1-a-b)]$. *E. F. Beckenbach.*

Integral Equations

Zanaboni, Osvaldo. Soluzione delle equazione integrale di Volterra avente un particolare nucleo in $x-y$. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) 3, 47–53 (1947).

The explicit solution of the integral equation

$$f(x)=\varphi(x)-\lambda\int_0^x\varphi(y)K(x-y)dy$$

is known when K is a finite sum of exponential functions [cf., for instance, Whittaker and Robinson, Calculus of Observations, Blackie, London, 1924, § 183]. The author obtains the corresponding solution for a more general type of kernels which are finite linear combinations of terms of the form $(x-y)^m e^{b_j(x-y)}$. The resolvent kernel is of the same form as K, the exponents can be obtained by solving an algebraic equation, and the coefficients from the decomposition of a rational function in partial fractions.

A. Erdélyi (Edinburgh).

Mandelbrojt, Szolem. Sur les noyaux singuliers symétriques. C. R. Acad. Sci. Paris 226, 1783–1785 (1948).

The author considers Carleman kernels C [Sur les Équations Intégrales Singulières à Noyau Réel et Symétrique, Uppsala, 1923; the definition is on p. 25]. In the definition of such kernels are involved certain singular lines $x=\xi_\nu$, $\eta=\xi_\nu$, where $\{\xi_\nu\}$ is a sequence with a finite number of limiting points at most. If, in addition, $|K(x,y)|$ has iterates of all orders belonging to L_2 (in y; $x\neq\xi_\nu$), then, according to Carleman, there is a certain inequality (1), involving the spectral function of $K(x,y)$, the iterates $K^{(j)}(x,y)$ $(j=2,\cdots,2m-1)$, $K^{(2m)}(x,x)$, $K^{(2m)}(y,y)$, the parameter λ and its angle. Let Δ be the square $\{a\leqq x, y\leqq b\}$ and F be the closure of the set of singular lines. The author uses the notation $(\alpha_n)=\sup\alpha_k$ for $k\geqq n$ and announces the following results for $K(x,y)\in C$ (in Δ).

(I) Suppose $|K(x,y)|$ has iterates of all orders in $\Delta-F$, belonging to L_2 (in y; if $x\neq\xi_\nu$) and that at a point (α,β) in $\Delta-F$ one has (2) $K^{(2m)}(\alpha,\beta)=0$ $(m\geqq1)$, $K^{(\lambda_n)}(\alpha,\beta)=0$ ($n\geqq1$; $\{\lambda_n\}$ a sequence of odd integers); let $M_m(x,y)=K^{(2m)}(x,x)K^{(2m)}(y,y)$, $p_m=p_m(x,y)=\log M_{m+1}(x,y)-\log M_m(x,y)$, $4L_n(x,y)=\sum_1^n\{p_m-p_{m-1}\}(m/n_m)$, where n_m is the number of $\lambda_n<2m$; if $\limsup n\lambda_n^{-1}>0$ and $\sum_1^\infty \exp(-L_n(\alpha,\beta))=\infty$, then $K^{(p)}(\alpha,\beta)=0$ for all $p\geqq2$.

(II) With the iterates of $|K(x,y)|$ as in (I), suppose that $K(x,y)$ is continuous in $\Delta-F$ and that

$$\int\int K(x,y)h(x)h(y)dxdy\geqq0$$

(for all real $h(x)$, belonging to L_2 and vanishing in the neighborhood of every ξ_ν); if (2) holds for a point (α,β) in $\Delta-F$ and if $\sum\exp[-L_n^*(\alpha,\beta)]=0$, where

$$4L_n^*=\sum_1^\infty[p_m-p_{m-1}][m/(m+n_m)],$$

then the conclusion of (I) holds. It is indicated that the results are proved on the basis of Carleman's inequality (1) and of a general theorem in the author's work [Ann. Sci. École Norm. Sup. (3) 63, 351–378 (1947); these Rev. 9, 229].

W. J. Trjitzinsky (Urbana, Ill.).

Parodi, Maurice. Sur une propriété d'une équation intégrale singulière. Bull. Sci. Math. (2) 71, 203–205 (1947).

It is known that, if φ, φ_1 are solutions of equations

$$L(\varphi)\equiv\varphi(x)-\lambda\int_0^x K(x-\tau)\varphi(\tau)d\tau=g(x),\quad L(\varphi)=g^{(1)}(x),$$

respectively, then $\varphi_1=\varphi^{(1)}$ if $\varphi(0)=0$. The author extends this result to singular equations

$$\varphi(t)-\lambda\int_0^\infty K(x)\varphi(\epsilon(x)+t)dx=g(t).$$

It is indicated that similar extensions can also be made for equations

$$\sum_0^p a_m\varphi^{(m)}(t)=g(t)+\lambda\int_a^b K(x)\varphi(\epsilon(x)+t)dx,$$

$$\int_0^\infty\varphi(x)K(x+t)dx=g(t).$$

W. J. Trjitzinsky (Urbana, Ill.).

Rabinovič, Yu. L. A proof of the closure of certain singular kernels. Doklady Akad. Nauk SSSR (N.S.) 61, 215–218 (1948). (Russian)

Under consideration are kernels $K(x,y)$ defined in $\Delta(a\leqq x, y\leqq b)$ and representable by finite sums of terms of form $|x-y|^\alpha H(x,y)$, $|x-y|^\alpha\log|x-y|H(x,y)$, $\delta(x,y)|x-y|^\alpha H(x,y)$, $\delta(x,y)|x-y|^\alpha\log|x-y|H(x,y)$, where $\delta(x,y)=1$ for $y<x$, $\delta(x,y)=-1$ for $y>x$; H and its partial derivatives of all orders are continuous in Δ. Let $D^j=\partial^j/\partial x^j$. Assuming that $D^mK>0$ (in Δ) and

$$D^m\int_a^b K(x,y)dy<0$$

(for $a\leqq x\leqq b$), the author proves the following. (I) If $|D^mK|<M|x-y|^{-\alpha}$ $(1\leqq\alpha\leqq2)$ and

$$J_m\equiv\int_a^b A_m dy=\int_a^b K dy=\infty$$

for $x=a$ and $x=b$, where $A_m=D^{m-2}K-(x-y)D^{m-1}K$, then the kernel $K(x,y)$ is closed in the class of Lipschitz functions. (II) Suppose $|D^{m-1}K|<M|x-y|^{-\alpha}$, $|D^mK|<M|x-y|^{-\alpha-1}$ $(1\leqq\alpha\leqq2)$, while the J_m and the $J_m'=dJ_m/dx$ are ∞ for $x=a$ and $x=b$; if the order of infinity of J_m is higher than that of J_m', then $K(x,y)$ is closed in the class of functions having a Lipschitz derivative.

These results are proved with the aid of the following lemma. If u is Lipschitz, $D^mK=H(x,y)|x-y|^{-\alpha}$ $(1\leqq\alpha\leqq2)$ and H is bounded, then

$$D^m\int_a^b K(x,y)u(y)dy=\int_a^b D^mK\cdot[u(y)-u(x)]dy+u(x)D^m\int_a^b K(x,y)dy;$$

a similar (but more complex) formula holds when u has a Lipschitz derivative, while $D^{m-1}K=H_1(x,y)|x-y|^{-\alpha}$, $D^mK=H_2(x,y)|x-y|^{-\alpha-1}$ ($1\leqq\alpha\leqq2$; H_i bounded).

W. J. Trjitzinsky (Urbana, Ill.).

Vaĭnberg, M. M. The existence of characteristic functions for a certain system of nonlinear integral equations. Doklady Akad. Nauk SSSR (N.S.) 61, 965–968 (1948). (Russian)

The author studies the system

$$\text{(1)}\qquad \mu_i u_i(x)=\int_B K_i(x,y)g_i(u_1(y),\cdots,u_n(y),y)dy$$

$(i=1,\cdots,n)$; it is assumed that the $K_i(x,y)$ are symmetric, positive, L_2 in (x,y), while $g_i=g_i(u_1,\cdots,u_n,x)=\partial G(u_1,\cdots,u_n,x)/\partial u_i$, $g_i(0,\cdots,0,x)=G(0,\cdots,0,x)=0$ (in B); B a bounded domain in Euclidean space. If for

$\mu_i = \mu_i^{(0)}$ (1) has a solution $u_i = \psi_i$ ($\psi_i \neq 0$ on a set of positive measure), then $(\mu_1^{(0)}, \cdots, \mu_n^{(0)})$, $(\psi_1, \cdots, \psi_n)$ are called characteristic values and characteristic functions of (1), respectively. The following is proved. If $\{\varphi_k\}$ is an orthonormal system in L_2, while $0 \leqq \lambda_1 \leqq \lambda_2 \leqq \cdots$, then the $\omega_m(x) = \sum_1^m \lambda_k^{-\frac{1}{2}} \zeta_k^{(m)} \varphi_k(x)$ $(m=1, 2, \cdots; \sum_1^m \zeta_k^{(m)2} \leqq c^2)$ are compact in L_2. The system (1) has at least a denumerable infinity of characteristic elements, belonging to $L_{2,n}$ and converging in mean square to zero, if the set of functions g_i constitutes a continuous operator in $L_{2,n}$. The system (1) has at least a denumerable infinity of characteristic elements, belonging to $L_{2,n}$ and converging in the mean square to zero, if the kernels $K_i(x, y)$ are symmetric, positive and belong to L_2, while every g_i satisfies (A_1) or (A_2); (A_1) g_i is continuous in $(u_1, \cdots, u_n)$, is measurable in x (in B) and satisfies $|g_i(u_1(x), \cdots, u_n(x), x)| \leqq L_i(x) \epsilon L_2$ or

$$|g_i| \leqq a_i(x) + b_i \sum_1^n |u_k(x)|^{p_{i,k}},$$

where $a_i(x) \epsilon L_{2+\alpha}$; $\alpha > 0$, $a_i(x) \geqq 0$, $b_i \geqq 0$, $0 \leqq p_{i,k} < 1$; (A_2) g_i is Lipschitz in $(u_1, \cdots, u_n, x)$.

These results are based, in part, on the work of Niemytzki [Rec. Math. [Mat. Sbornik] **41**, 421–458 (1934)].

W. J. Trjitzinsky (Urbana, Ill.).

Mihlin, S. G. Singular integral equations. Uspehi Matem. Nauk (N.S.) **3**, no. 3(25), 29–112 (1948). (Russian)

The subject of singular integral equations (in the sense of principal values) had its beginnings in some of the work of Poincaré and of Hilbert, was later continued by F. Noether [Math. Ann. **82**, 42–63 (1920)], Carleman [Ark. Mat. Astr. Fys. **16**, no. 26 (1922)] and subsequently by Mushelishvili, Mihlin, Giraud [Ann. Sci. École Norm. Sup. (3) **51**, 251–372 (1934); **56**, 119–172 (1939); these Rev. **1**, 145; and some other papers], Tricomi [for example, Math. Z. **27**, 87–133 (1927)], to mention just a few. In pages 32–79 the author gives a systematic exposition of the essential features of the theory of singular integral equations in the complex plane, with integrations in the sense of Cauchy principal values along sufficiently smooth curves, open or closed and having no intersections. Much of this work has to do with the reduction to regular Fredholm equations of the second kind. One approach consists in the elaboration of Carleman's idea of relating singular integral equations to boundary value problems of Hilbert-Riemann type; this is combined with certain other ideas. Another approach [Mihlin] consists in finding operators whose application to the given equations renders them regular. Developments relating to singular integral equations involving multi-dimensional integrals are given in pp. 79–111. Here the notable work is due to Mihlin, Giraud (some of their contributions are mutually supplementary) and to Tricomi. The method is that of constructing suitable operators, whose application to the given singular integral equations regularizes them. The theory in the two-dimensional Euclidean plane has been developed by Mihlin (this involves use of Fourier series to represent the kernels). The extension to higher dimensional spaces involves expansions into spherical harmonics and has been made possible by an extension by Giraud [C. R. Acad. Sci. Paris **202**, 2124–2127 (1936)] of Mihlin's work on composition of singular integrals. It is shown how to treat singular integral equations with integrals extended over an m-dimensional manifold Ω, without edges and imbedded in a Euclidean space of p $(>m)$ dimensions. Some extensions to Hilbert space are given.

W. J. Trjitzinsky (Urbana, Ill.).

Šerman, D. I. On methods of solving certain singular integral equations. Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 423–452 (1948). (Russian)

This is a study of systems of the form

$$(1) \quad \sum_{j=1}^{2} \left\{ a_{kj}(t_0)\omega_j(t_0) + b_{kj}(t_0)(\pi i)^{-1} \int_L (t-t_0)^{-1}\omega_j(t)dt \right\} = f_k(t_0)$$

$(k=1, 2)$, where L is a simple, closed, "smooth" curve (in the complex plane of $z=x+iy$), bounding a finite simply-connected region S; the $\omega_j(t)$ are the unknowns and the a_{kj}, b_{kj}, f_k are assigned, suitably differentiable on L; the integrals are in the sense of principal values. On letting $c_{kj} = a_{kj} - b_{kj}$, $d_{kj} = a_{kj} + b_{kj}$, one forms determinants $\Delta_1 = |(c_{ij})|$, $\Delta_2 = |(d_{ij})|$. The extensive literature relating to equations of type (1), and of other similar types, is largely concerned with transformations into regular Fredholm equations of the second kind, when Δ_1, Δ_2 (or other analogous functions) are distinct from zero on L. One of the novel features of this work is that one of the determinants is allowed to vanish at some points of L (the other one is assumed not 0 on L). Specifically, it is assumed that $\Delta_1(t)$ has a simple zero at $t=\alpha$ (on L) and the coefficients are analytic at $t=\alpha$ (the latter condition can be lightened). It is shown that a reduction to regular Fredholm equations is possible and that (1) has a solution ω_j $(j=1, 2)$ continuous on L. Such results are extended to systems

$$(2) \quad \sum_{j=1}^{2} \left\{ a_{kj}(t_0)\omega_j(t_0) + (\pi i)^{-1} b_{kj}(t_0) \int_L (t-t_0)^{-1}\omega_j(t)dt + T_j^k \right\} = f_k(t_0)$$

$(k=1, 2)$, where $T_j^k = \int_L \omega_j(t) K_{kj}(t, t_0)dt$; here the $K_{kj}(t, t_0)$ (and the coefficients) are Hölder on L, in t, in t_0, and are analytic at the point $t_0 = \alpha$, at which Δ_1 has a zero of multiplicity m. The system (1) is also studied when the a_{kj}, b_{kj} are constants and L is an open arc. *W. J. Trjitzinsky.*

Sarmanov, O. V. On the rectification of correlation. Uspehi Matem. Nauk (N.S.) **3**, no. 5(27), 190–192 (1948). (Russian)

Summary of the author's dissertation. Cf. his papers in Doklady Akad. Nauk SSSR (N.S.) **58**, 745–747 (1947); **59**, 861–863, 1061–1064 (1948); **60**, 545–548 (1948); these Rev. **10**, 45; **9**, 442, 593.

Wallace, P. R. Angular distribution of neutrons inside a scattering and absorbing medium. Canadian J. Research. Sect. A. **26**, 99–114 (1948).

The solution of the equation of transfer in an infinite homogeneous medium for an arbitrary distribution of sources is known [cf., for example, G. Placzek, National Research Council of Canada, Division of Atomic Energy, Document no. **MT-4** (1943); these Rev. **9**, 190]. In terms of this solution the author obtains explicit expressions for the various moments (with respect to integrations over the angle) of the intensity. *S. Chandrasekhar* (Williams Bay, Wis.).

Placzek, G., and Volkoff, G. A theorem on neutron multiplication. Canadian J. Research. Sect. A. **25**, 276–292 (1947).

The principal theorem established in this paper concerns the asymptotic behavior of the inhomogeneous integral equation

$$(*) \qquad f(z) = QG_s(z) + k\int_{-\infty}^{\infty} F(|z-z'|)f(z')dz',$$

where Q and k ($\leqq 1$) are certain positive constants and $F(|z|)$ and $G_s(|z|)$ are known functions which fall off at least exponentially at large $|z|$ so that $F(|z|)e^{\eta z}$ and $G_s(|z|)e^{\eta z}$ are quadratically integrable in the whole interval $-\infty < z < \infty$ for $|\eta| < \eta_f$ and $|\eta| < \eta_s$, respectively (η_f and η_s being positive constants characteristic of $F(|z|)$ and $G_s(|z|)$). Further $\int_{-\infty}^{\infty} F(|z|)dz = 1$.

By applying to (*) a Fourier transformation the authors show that the solution of (*) can be expressed in the form

$$f(z) = (Q/2\pi)\int_{-\infty}^{\infty} \frac{\Gamma_s(\zeta)e^{-iz\zeta}}{1-k\Phi(\zeta)}d\zeta,$$

where $\Gamma_s(\zeta)$ and $\Phi(\zeta)$ are the Fourier transforms of $G_s(|z|)$ and $F(|z|)$, respectively. With the solution expressed in this form, it follows that the asymptotic behaviour of $f(z)$ for large $|z|$ is given by $Ce^{-\mu|z|}$ ($|z| \to \infty$), where μ is defined as the positive real root of $\Phi(i\mu) = k^{-1}$ and

$$C = Qk^{-1}\int_0^\infty G_s(|z|)\cosh \mu z\, dz \Big/ \int_0^\infty F(|z|)z \sinh \mu z\, dz.$$

Various special cases of this theorem are considered and illustrated. *S. Chandrasekhar* (Williams Bay, Wis.).

Rocard, Yves. Sur une propriété de certaines équations fonctionnelles. C. R. Acad. Sci. Paris **227**, 502–504 (1948).

The author considers (*) $y'(t) + a\int_0^\infty f(\tau)y(t-\tau)d\tau = 0$ for real positive a and τ; $f(\tau)$ is a distribution function such that $\int_0^\infty f(\tau)d\tau = 1$ and τ_1 is the first moment. If $g(\tau)$ is a similar function with first moment also τ_1 and if $\int_0^{\tau_1}(\tau_1-\tau)g(\tau)d\tau > \int_0^{\tau_1}(\tau_1-\tau)f(\tau)d\tau$ then the equation (*) with $g(\tau)$ in place of $f(\tau)$ is more stable than (*) with $f(\tau)$ in the following sense. If a_f is such that for $a < a_f$ the equation has only stable solutions while for $a > a_f$ it may have unstable solutions, then $a_g > a_f$. No proof is given.

N. Levinson (Copenhagen).

Functional Analysis, Ergodic Theory

Doss, Raouf. On uniformly continuous functions in metrizable spaces. Proc. Math. Phys. Soc. Egypt **3** (1947), 1–6 (1948).

The following characterizations are shown. (I) In a metric space, every (real-valued continuous) function is uniformly continuous if and only if every sequence x_n, for which there exists another, y_n, disjoint from the former and such that $(x_n, y_n) \to 0$, has a convergent subsequence. (II) In a metrizable space E, every function is uniformly continuous relavtie to every metric for E if and only if E is compact. (III) A metrizable space E has a metric relative to which every function is uniformly continuous if and only if every sequence of nonisolated points has a convergent subsequence.

R. Arens (Los Angeles, Calif.).

Menger, Karl. Generalized vector spaces. I. The structure of finite-dimensional spaces. Canadian J. Math. **1**, 94–104 (1949).

A vector space V with real multipliers and a generalized "norm" $|v|$, which is a real-valued functional satisfying only the two conditions (1) $|\alpha v| = \alpha|v|$ for $\alpha > 0$ and (2) $|v+w| \leqq |v| + |w|$, is called a generalized vector space. A vector v in V is called positive, negative or zero according to whether $|v|$ is positive, negative or zero. The author proposes to study these spaces in a series of papers, of which this first paper is concerned with the structure of finite dimensional spaces, while infinitely dimensional spaces and applications to analysis, etc., are to be dealt with in subsequent articles. The principal results are as follows. It is proved that every V is the join, $V = V_d + V'$, of two subspaces, where V_d is the subspace of V containing all vectors v such that $|v| = |-v| = 0$. In case V has a finite dimension, it is shown further that $V = V_d + V_p + L$, where V_d has the same meaning as before, while V_p contains (besides the origin) only positive vectors, and L is either a single line containing a nonpositive vector v_0 such that $-v_0$ is positive, or else L contains only the origin. *D. H. Hyers.*

Julia, Gaston. Détermination de toutes les racines carrées d'un opérateur hermitien borné quelconque. C. R. Acad. Sci. Paris **227**, 792–794 (1948).

Julia, Gaston. Détermination de toutes les racines carrées d'un opérateur hermitien borné quelconque (2e méthode). C. R. Acad. Sci. Paris **227**, 931–933 (1948).

Extending his earlier results [same C. R. **222**, 707–709, 829–832, 1019–1022, 1061–1063 (1946); these Rev. **7**, 452, 453], the author determines (in terms of certain invariantly defined linear manifolds) all (not necessarily Hermitian) square roots of all (not necessarily definite) bounded Hermitian operators. *P. R. Halmos* (Chicago, Ill.).

Korenblyum, B. I., Kreĭn, S. G., and Levin, B. Ya. On certain nonlinear questions of the theory of singular integrals. Doklady Akad. Nauk SSSR (N.S.) **62**, 17–20 (1948). (Russian)

Let E be a separable Banach space, and $\bar{E}$ the space conjugate to E. Let $C([0, 1]; E)$ denote the space of all functions $\hat{f} = f_\tau$ ($0 \leqq \tau \leqq 1$), having values in E, and strongly continuous on the segment $[0, 1]$. The norm $\|\hat{f}\|$ is defined as $\max_{0\leqq\tau\leqq 1}\|f_\tau\|$. (1) Modifying the result of Gowurin [Fund. Math. **27**, 254–268 (1936)] the authors state that every linear functional in $C([0, 1]; E)$ is representable in the form (*) $F(\hat{f}) = \int_0^1 \varphi_\tau(f_\tau)d\sigma(\tau)$, where $\sigma(\tau)$ is nondecreasing in $[0, 1]$ and φ_τ a B-measurable abstract function with values in $\bar{E}$, satisfying $\|\varphi_\tau\| = 1$ ($0 \leqq \tau \leqq 1$). Conversely every expression (*) is a linear functional in $C([0, 1]; E)$ with norm $\|F\| = \mathrm{var}_{0\leqq\tau\leqq 1}\,\sigma(\tau)$. The representation (*) is unique except for the normalization of $\sigma(\tau)$ and the values of φ_τ on a set of measure 0 with respect to $\sigma(\tau)$. (2) Let L_0^p ($p > 1$) be the space of all functions $f(x) \in L^p(0, 1)$ satisfying $\lim_{h\to 0} h^{-1}\int_0^h |f(x)|^p dx = 0$, the norm $\|f\|_0^p$ being defined as $\max_{0<h\leqq 1}[h^{-1}\int_0^h |f(x)|^p dx]^{1/p}$. Then the general linear functional in L_0^p is of the form (**) $F(f) = \int_0^1 F(x)f(x)dx$, where the function $F(x)$ has the following property: the maximal convex function $\Phi(x)$ majorized on $(0, 1)$ by $\int_x^1 |F(t)|^q dt$ ($p^{-1} + q^{-1} = 1$) satisfies $\int_0^1 [-\Phi'(x)]^{1/q}dx < \infty$. Conversely every $F(x)$ having this property generates a functional (**) with norm $\|F\| = \int_0^1 [-\Phi'(x)]^{1/q}dx$. This is obtained as a special case of a general result concerning functionals associated with kernels $\theta(x, \tau)$ defined in the square $0 \leqq x, \tau \leqq 1$. Results of similar type are obtained for sequences of functionals. *A. Zygmund* (Chicago, Ill.).

Dixmier, J. Sur un théorème de Banach. Duke Math. J. **15**, 1057–1071 (1948).

Let E be a Banach space, let E' be its conjugate and let V be a subspace of E'. Let $s \geqq 0$ be the least upper bound of numbers k satisfying the inequality $k\|x\| \leqq \sup |f(x)|/\|f\|$ for $f \in V$, $f \neq 0$. Let $r \geqq 0$ be the least upper bound of the numbers R such that the intersection with V of the sphere

about the origin with unit radius is weakly dense in the sphere with the same center and radius R. It is easy to see that when E is separable then $r>0$ if and only if E' coincides with the set V^1 of all weak limits of sequences of members of V. The theorem of Banach referred to in the title asserts that if E is separable then $V^1=E'$ if and only if $s>0$. The author generalizes and sharpens this result by showing that for arbitrary E and V it is true that $r=s$. This number he calls the characteristic of V. Two further equivalent definitions of the characteristic of V are given and in the latter half of the paper they are applied to the study of the nature of conjugate spaces. A central notion here is that of a minimal subspace, this being defined as a subspace of E' which is strongly closed, weakly dense and minimal among subspaces with these properties. Some typical theorems are as follows. If V is minimal then its characteristic is greater than zero; E is linearly homeomorphic to the conjugate of some space if and only if E' contains a minimal subspace; E is isometric to the conjugate of some space if and only if E' contains a minimal subspace of characteristic one. *G. W. Mackey* (Cambridge, Mass.).

Dixmier, Jacques. Fonctionnelles linéaires sur l'ensemble des opérateurs bornés d'un espace hilbertien. C. R. Acad. Sci. Paris **227**, 948–950 (1948).

Let B be the Banach space of all bounded linear transformations on a Hilbert space. Let $\mathfrak{J}$ be the subspace of completely continuous transformations. The author observes first that there is a natural isometric linear mapping of the second conjugate $\mathfrak{J}''$ of $\mathfrak{J}$ onto all of B and hence a natural map of $\mathfrak{J}'$ in B'. Next the image of $\mathfrak{J}'$ in B' is characterized in various ways. In particular, it is the set of all members of B' continuous with respect to the so-called "strongest" topology in B. Then a discussion of the relationship between various types of closure for subspaces and subalgebras of B is given. Finally several applications are indicated. Some of these are to von Neumann and Murray's theory of rings of operators. In particular, it is asserted that the property of being a subring of a ring M is "purely algebraical" even if M is not in a finite class, thus improving a result of Murray and von Neumann [Ann. of Math. (2) **44**, 716–808 (1943); these Rev. **5**, 101]. In another a question raised by the reviewer [Trans. Amer. Math. Soc. **57**, 155–207 (1945), problem 12; these Rev. **6**, 274] is answered in the negative. No proofs are given. *G. W. Mackey.*

Grinblyum, M. M. On a property of a basis. Doklady Akad. Nauk SSSR (N.S.) **59**, 9–11 (1948). (Russian)

This note characterizes Schauder bases among fundamental sets. For $\{x_i\}$ fundamental in a separable Banach space X let A be the space of all numerical sequences $\{a_i\}$ for which $\sum a_i x_i$ converges in X, and assume there is a norm in A making A a Banach space with the unit vectors $(0, 0, \cdots, 0, 1, 0, \cdots)$ as a basis. Let B be the space of all numerical sequences $\{b_i\}$ such that $\sum a_i b_i$ converges for every $\{a_i\}$ in A. Then $\{x_i\}$ is a basis if and only if for each $\{b_i\}$ in B there is an F in X^* with $F(x_i)=b_i$, $i=1, 2, \cdots$. *M. M. Day* (Princeton, N. J.).

Titov, N. S. Concerning various forms of convergence of elements or linear operators in Banach spaces. Uspehi Matem. Nauk (N.S.) **1**, no. 5–6(15–16), 228–229 (1946). (Russian)

Let E, E', and E'' be Banach spaces and let (E, E') be the space of all linear operators from E into E'. For x_n and x in E say that x_n converges (E') to x if $\lim_n \|u(x_n-x)\|=0$ for every u in (E, E'); for u_n and u in (E, E') say that u_n converges (E'') to u if $u_n(x)$ converges (E'') to $u(x)$ for every x in E. This note gives some examples where these differ from the weak and strong convergence in their respective spaces. *M. M. Day* (Princeton, N. J.).

Tatarkiewicz, Kyrzysztof. Sur la convexité des sphères et sur l'approximation dans les espaces de Banach. C. R. Acad. Sci. Paris **227**, 1332–1333 (1948).

Let B be a Banach space and M a finite-dimensional subspace of B. The author has observed the well-known facts that (1) the set E_x of points of M at minimal distance from an x of B is nonempty and depends continuously on x; (2) if B is strictly convex, then E_x contains only one point. *M. M. Day* (Princeton, N. J.).

Alexiewicz, A., et Orlicz, W. Sur la continuité et la classification de Baire des fonctions abstraites. Fund. Math. **35**, 105–126 (1948).

This is a paper on Banach-valued functions, but it ends up with the following theorem on numerical functions. For $p\geqq 1$, if for a sequence of real functions $f_n(u)$ on an interval we everywhere have $\sum_n |f_n(u)|^p<\infty$, then for a set of second category we also have $\lim_{u\to u_0} \sum_n |f_n(u)-f_n(u_0)|^p=0$.

If X is a Banach space, and Y the space of functionals on it, let Y_0 be a (nonclosed) linear subspace which is "weakly dense in Y" in the following sense: for $x_0 \epsilon X$, $\epsilon>0$, there exists a ξ in Y_0 such that $|\xi(x)|\leqq\|x\|$, x in B, $|\xi(x_0)|\geqq\|x_0\|-\epsilon$. Now, let $x(u)$ be a function from an interval on some more general metric space to a separable X. If it is weakly continuous in the sense of $\xi(x(u))\to\xi(x(u_0))$ for ξ in Y_0, then strongly it belongs to the first Baire class. More generally if it is weakly in the α-class it is strongly in the $(\alpha+1)$-class. But if its range of values is totally bounded, it remains in the α-class strongly. *S. Bochner* (Princeton, N. J.).

Shimoda, Isae. On power series in abstract spaces. Math. Japonicae **1**, 69–73 (1948).

Let $U_n(x)$ be a homogeneous polynomial of degree n, defined on E and having values in E', where E and E' are complex Banach spaces. This paper is concerned with the determination of the largest open sphere about the origin in E in which the power series $\sum_{n=0}^{\infty} U_n(x)$ defines an analytic function, i.e., a function which is continuous and differentiable in the sense of Gâteaux. The radius of this sphere is called the radius of analyticity of the series. To state the first result we write $m_n(G)=\sup \|U_n(x)\|$ for $x \epsilon G$, where G is any set in E. Let K be the family of all compact subsets of the surface of the unit sphere in E. Then the radius of analyticity τ of the series is given by

$$1/\tau=\sup_{G\epsilon K} \limsup_{n\to\infty} [m_n(G)]^{1/n}.$$

[Reviewer's remark. The same result holds if K is the family of all compact subsets of the solid unit sphere.] The second result of the paper is that

$$1/\tau=\sup_{\|x\|=1} \limsup_{n\to\infty} \|U_n(x)\|^{1/n}.$$

This theorem is contained in the exposition of the theory of analytic functions in E. Hille's recent book [Functional Analysis and Semi-Groups, Amer. Math. Soc. Colloquium Publ., v. 31, New York, 1948, in particular, theorems 4.7.4 and 4.7.6; these Rev. **9**, 594]. Shimoda's proof is based on the theorem of Zorn to the effect that if $f(x)$ is Gâteaux-differentiable and continuous in the sense of Baire on an open set then it is analytic there [Ann. of Math. (2) **46**,

585–593 (1945), in particular, p. 591; these Rev. 7, 251]. The principal problem is to show that the series defines a Gâteaux-differentiable function when $\|x\|<r$. The demonstration uses Poisson's integral and subharmonic functions in a manner reminiscent of the proof of the Osgood-Hartogs theorem on functions of several complex variables.

A. E. Taylor (Los Angeles, Calif.).

Stone, M. H. On a theorem of Pólya. J. Indian Math. Soc. (N.S.) 12, 1–7 (1948).

The author proves in a simple manner a generalization of a theorem of Pólya which has been used in the study of quasi-nilpotent elements of a Banach algebra. Let f be an entire function from the complex field to a complex Banach space B. Then, if $\|f(re^{i\theta})\|=O(|r\sec\theta|^N)$, f is a polynomial of degree not greater than N. The theorem of Pólya follows; if f is entire, and $g(\mu)=f([1+\mu]/[1-\mu])$ where $g(\mu)=\sum b_n\mu^n$, convergent for $|\mu|<1$, and $g(\mu)=\sum c_n\mu^{-n}$, convergent for $|\mu|>1$, and if $\|b_n\|$ and $\|C_n\|$ are each $O(n^N)$, then f is a polynomial of degree not greater than N. If B is a Banach algebra, and x is a quasi-nilpotent element, then a necessary and sufficient condition that $x^{N+1}=0$ is that $\|(e-x)^n\|=O(|n|^N)$ for $n=0, \pm 1, \pm 2, \cdots$. If $N\geqq 1$, then $x^{N+1}=0$ may be replaced by $x^N=0$ if $O(|n|^N)$ is replaced by $o(|n|^N)$ [Hille, Proc. Nat. Acad. Sci. U. S. A. 30, 58–60 (1944); Functional Analysis and Semi-groups, Amer. Math. Soc. Colloquium Publ., v. 31, New York, 1948, p. 493; these Rev. 5, 189; 9, 594]. By an application of the previous result this is proved in the present paper in a generalized form where it is merely assumed that B has a product and an identity obeying conditions which guarantee that the closure of the ring of polynomials in x forms a Banach algebra, for each $x\epsilon B$.

R. C. Buck.

Naĭmark, M. A. Rings with involutions. Uspehi Matem. Nauk (N.S.) 3, no. 5(27), 52–145 (1948). (Russian)

A "ring with involution" is a linear algebra R (with identity element e) over the field of complex numbers in which there is defined a mapping $x\rightarrow x^*$ of R onto itself such that $(x^*)^*=x$, $(\lambda x)^*=\bar{\lambda}x^*$, $(xy)^*=y^*x^*$, $(x+y)^*=x^*+y^*$ and which is also a normed linear space whose norm satisfies $\|e\|=1$, $\|xy\|\leqq\|x\|\,\|y\|$, $\|x^*\|=\|x\|$. The present paper contains a detailed study of such rings and represents mainly joint work of I. M. Gelfand and the author. Much of the material has appeared previously in one form or another [Gelfand and Naĭmark, Rec. Math. [Mat. Sbornik] N.S. 12(54), 197–217 (1943); C. R. (Doklady) Acad. Sci. URSS (N.S.) 55, 567–570 (1947); Izvestiya Akad. Nauk SSSR. Ser. Mat. 11, 411–504 (1947); 12, 445–480 (1948); these Rev. 5, 147; 8, 563; 9, 495; 10, 199]. The discussion falls into three chapters, the first of which contains more or less well-known introductory material.

Chapter II is devoted to representation theory. A representation of R is a homomorphism $x\rightarrow A_x$ of R into the ring of bounded operators on a Hilbert space $\mathfrak{H}$ such that $A_{x^*}=A_x^*$, where A_x^* is the operator adjoint to A_x. If R is complete, then every representation is continuous and $\|A_x\|\leqq\|x\|$. A representation is called "irreducible" provided (0) and $\mathfrak{H}$ are the only invariant subspaces. It is called "cyclic" provided there exists a "cyclic" vector $\xi_0\epsilon\mathfrak{H}$ such that the vectors $A_x\xi_0$ are dense in $\mathfrak{H}$. Two representations $x\rightarrow A_x$, $x\rightarrow B_x$ on $\mathfrak{H}$, $\mathfrak{H}'$, respectively, are called "equivalent" provided there exists an isometry U of $\mathfrak{H}$ on $\mathfrak{H}'$ such that $B_xU=UA_x$. The following generalization of a lemma of Schur is proved. Let T be an operator on $\mathfrak{H}$ to $\mathfrak{H}'$ which is nonsingular in the sense that it is closed, its domain of definition is dense in $\mathfrak{H}$ and the zero manifolds of T and T^* reduce to (0); then $B_xT\subset TA_x$ implies equivalence of the two representations.

An important tool in the study of representations is the positive functional: i.e., a linear functional $f(x)$ on R such that $f(x^*x)\geqq 0$ for all x. The class of all positive functionals on R is denoted by P. Each $f\epsilon P$ automatically satisfies $f(x^*)=\overline{f(x)}$ and, if R is complete, is bounded with bound $f(e)$. The set I_0 of all $x\epsilon R$ such that $f(x^*x)=0$ for every $f\epsilon P$ is a 2-sided symmetric (i.e., $I_0^*=I_0$) ideal in R. If $I_0=(0)$, then R is said to be "reduced." The ring $R_0=R/I_0$ is reduced. If $x\rightarrow A_x$ is a representation of R on $\mathfrak{H}$ and $\xi_0\epsilon\mathfrak{H}$, then $f(x)\equiv(A_x\xi_0, \xi_0)$ defines an element of P. If $x\rightarrow A_x$, $x\rightarrow B_x$ are cyclic representations with cyclic vectors ξ_0, ξ_0' and if $(A_x\xi_0, \xi_0)\equiv(B_x\xi_0', \xi_0')$, then the representations are equivalent. If R is complete and $f\epsilon P$, then there exists a cyclic representation $x\rightarrow A_x$ such that $f(x)\equiv(A_x\xi_0, \xi_0)$. If R is complete and H is the real B-space of all $x\epsilon R$ such that $x^*=x$, then the set K of all $f\epsilon P$ such that $f(e)=1$ is a closed convex subset of the unit sphere in the space conjugate to H. The extreme points of K determine the irreducible representations of R. This fact implies that, if R is complete and reduced, then its irreducible representations constitute a complete family. These results parallel those obtained by I. E. Segal for algebras of operators on Hilbert space [Bull. Amer. Math. Soc. 53, 73–88 (1947); these Rev. 8, 520].

The norm in R is said to be regular provided every $f\epsilon P$ can be extended as a positive functional on the completion of R. If R is complete and reduced, then it possesses a minimal regular norm. Moreover, the completion of R under this minimal norm is norm and *-isomorphic to a ring of bounded operators on Hilbert space.

Chapter III contains a discussion of certain special rings. Let R be complete and denote by $R_0=R/I_0$ its associated reduced ring. Denote by $\tilde{R}$ the completion of R_0 under its minimal regular norm. If $R=\tilde{R}$, then R is said to be "*-complete." If R is complete and commutative and $\mathfrak{M}_0$ is its set of symmetric maximal ideals, then $\tilde{R}$ is norm and *-isomorphic to the ring $C(\mathfrak{M}_0)$ of all continuous functions on $\mathfrak{M}_0$. Also, every positive functional on R has the form $f(x)=\int_{\mathfrak{M}_0}x(M)d\sigma(\Delta)$, where $\sigma(\Delta)$ is a nonnegative completely additive function of Borel subsets of $\mathfrak{M}_0$. Call R a "*-ring" provided $\|x^*x\|=\|x\|^2$. Every *-isomorphism of one complete *-ring into a second is norm preserving. Call R "symmetric" provided $(e+x^*x)^{-1}$ exists in R for every x. Every *-complete ring is symmetric. A commutative and complete R is symmetric if, and only if, each of its maximal ideals is symmetric (i.e., $M^*=M$). If R_1 is a closed *-subring of a complete symmetric ring R, then every (irreducible) representation of R_1 can, in a natural way, be extended to a (irreducible) representation of R.

An important type of ring with involution is the group ring of a locally compact group. Known results relating unitary representations of a group and representations of its group ring are obtained. The cases of the group of linear transformations on the line and the complex unimodular group G_2 in two dimensions (and hence the Lorentz group) are studied in some detail [see the second and third references cited above]. The group ring of G_2 is an example of a group ring which is not symmetric.

The paper closes with a study of representations of the ring B of all bounded operators on (separable) Hilbert space. Every such representation is a direct sum of identity representations $A\rightarrow A$ and a representation of B/I, where I is the

(unique) closed 2-sided ideal in B consisting of all completely continuous operators. *C. E. Rickart.*

Šreĭder, Yu. A. The structure of maximal ideals in rings of completely additive measures. Doklady Akad. Nauk SSSR (N.S.) **63**, 359–361 (1948). (Russian)

A summary is given, with some indications of proofs, of a study [a continuation of earlier work, notably that of Gel'fand, Raĭkov and Šilov, Uspehi Matem. Nauk (N.S.) 1, no. 2(12), 48–146 (1946); these Rev. **10**, 258] of the maximal ideals in the algebra A of all complex-valued countably-additive functions on the Borel subsets of the reals, multiplication being convolution. Every such ideal corresponds to a unique homomorphism of A onto the algebra of complex numbers, and every such homomorphism is stated to have the form $\varphi\rightarrow M(\varphi)=\int_{-\infty}^{\infty}m_\varphi(t)d(t)$ where $m(t)$ is a "generalized" function [whose precise character is not explained] which for each $\varphi\epsilon A$ is to be representable by a bounded φ-measurable function of t, and which satisfies the equation $m(s)m(t)=m(s+t)$ for almost all (s, t) relative to the product measure $\varphi\times\varphi$. It is stated that there exist homomorphisms M with the property that $\overline{M(\varphi)}\neq M(\varphi^*)$, where $\varphi^*(E)=\overline{\varphi(-E)}$ for any Borel set E, and that from this the following result of Wiener and Pitt [Duke Math. J. **4**, 420–436 (1938)] is deducible: there exists an element of A whose Fourier-Stieltjes transform is bounded away from zero and such that the reciprocal of the transform is not the transform of any element of A. An m_φ with the above property is called a "generalized character," and is said to induce a decomposition of A into a direct sum of the subalgebra of ψ such that $m_\psi(t)\neq 0$ almost everywhere and the ideal of φ such that $m_\varphi=0$; and the class of all such decompositions can be characterized by intrinsic properties.

I. E. Segal (Chicago, Ill.).

Grabar', M. The representation of dynamical systems as systems of solutions of differential equations. Doklady Akad. Nauk SSSR (N.S.) **61**, 433–436 (1948). (Russian)

Let R be a compact metric space and R_t a dynamical system (flow) in R, with at most one fixed point. Denote by $p_t(-\infty<t<\infty)$ the points of the trajectory of P. It follows from results of Bebutoff [Bull. Math. Univ. Moscow **2**, no. 3 (1939); C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 904–906 (1940); these Rev. **1**, 281; **2**, 225] that there is a continuous mapping ψ of R into the real numbers such that, if $p\neq q$, the two real functions $\psi_p(t)=\psi(p_t)$ and $\psi_q(t)=\psi(q_t)$ are not identical. Thus ψ induces a one-to-one map of the flow in R into the translational flow in the space of continuous real functions of t. Now let L_m^2 be the Hilbert space of square-integrable real functions of t over the range $J:-\infty<t<\infty$, relative to a completely additive measure m such that (1) $m(J)$ is finite; (2) $m(I)>0$ for each interval I; (3) for each measurable set B, $m(B)=\inf m(G)$ for all open sets $G\supset B$. It is then shown that the map Φ of R such that $p\rightarrow\phi_p(t)=e^{-t^2}\int_0^t\psi_p(t)dt$ is a homeomorphism of R into L_m^2 such that the trajectories of R_t are mapped onto the solutions of differential equations $dx_i/dt=f_i(x_1, \cdots, x_n, \cdots)$, in terms of the coordinates x_i in the Hilbert space L_m^2. More precisely, if R is regarded as imbedded in L_m^2, along each trajectory $x(t)$ the difference quotient $[x(t+h)-x(t)]/h$ converges strongly, as $h\rightarrow 0$, to an element $x'(t)$ of L_m^2, and this derivative has a value $F(x)$, where F is a continuous operator in L_m^2. It is remarked that the theorem can be extended to the case of a space R which is locally compact and has a countable base, provided the flow has no singular point. *W. Kaplan* (Ann Arbor, Mich.).

Budak, B. M. Dispersive dynamical systems. Vestnik Moskov. Univ. **1947**, no. 8, 135–137 (1947). (Russian)

This is a short summary of the author's Moscow thesis. Let R_n be Euclidean n-space. With a differential system $dx/dt=X(x)$, where x and X are n-vectors, there is associated a family of mappings $f(p, t)$: $R_n\rightarrow R_n$; one and only one mapping corresponds to a given point t of the real line L: $-\infty\leqq t<+\infty$. Let now R be any metric space, A and B two subsets of R, $U(B, \epsilon)$ an ϵ-neighborhood of B, and define

$$\vec{\alpha}(A, B)=\inf\{\epsilon|A\subset U(B, \epsilon)\};$$

$$\alpha(A, B)=\max\{\vec{\alpha}(A, B), \vec{\alpha}(B, A)\}.$$

A dispersive dynamical system is a family of transformations (not necessarily single-valued) $f(p, t):R\rightarrow R$ defined for each $t\epsilon L$ and satisfying the following conditions: (I) $f(p, 0)$ is the identity; (II) $f(p, t)$ is a nonempty compactum; (III) $q\epsilon f(p, t)$ implies $p\epsilon f(q, -t)$; (IV) $f(f(p, t_1), t_2)=f(p, t_1+t_2)$; (V) if $t\rightarrow t_0$ then $\alpha(f(p, t), f(p, t_0))\rightarrow 0$ whatever $t_0\epsilon L$ and $p\epsilon R$; (VI) if $p\rightarrow p_0$ and $t\rightarrow t_0$ then $\vec{\alpha}(f(p, t), f(p_0, t_0))\rightarrow 0$ whatever $p_0\epsilon R$ and $t_0\epsilon L$. A motion through p is a mapping $\varphi_p(t):L\rightarrow R$ such that (a) $\varphi_p(0)=p$; (b) if $t^*<t^{**}$ then $\varphi_p(t^{**})\epsilon f(\varphi_p(t^*), t^{**}-t^*)$. The set $\varphi_p(L)$ is the trajectory through p. The motion $\varphi_p^*(t)$ is dynamically congruent to $\varphi_p(t)$ whenever there is a t^* such that $\varphi_p^*(t)=\varphi_p(t^*+t)$ for all $t\epsilon L$. The ordinary dynamical systems in R_n are dispersive. The author gives criteria for the existence of noncongruent motions on one and the same trajectory and gives examples of such noncongruent motions. Many of the well-known results of G. D. Birkhoff, among others, are extended to dispersive systems. Properties of α- and ω-limiting sets are investigated. *S. Lefschetz* (Princeton, N. J.).

Rauch, H. Ernest. Généralisation d'une proposition de Hardy et Littlewood et de théorèmes ergodiques qui s'y rattachent. C. R. Acad. Sci. Paris **227**, 887–889 (1948).

Let S denote the $(2n-1)$-dimensional unit sphere, $S_Q(\rho)$ the set of points on S with spherical distance from Q smaller than ρ, $V(\rho)$ the volume of $S_Q(\rho)$ and dV the volume element on S. If $f(P)$ is of class L on S and σ denotes the set of points Q for which

$$\sup_{0\leqq\rho<r}\frac{1}{V(\rho)}\int_{S_Q(\rho)}|f(P)|dV>\alpha,$$

then $m(\sigma)\leqq 3^{2n-1}\alpha^{-1}\int_S|f(P)|dV$, where $m(\sigma)$ denotes exterior measure. The author announces two applications of this result. Let first $f(z_1, \cdots, z_m)$ be a function regular in the unit sphere. In polar coordinates we write the function $f(s, P)$, P denoting a point of the unit sphere. If for some $\lambda>0$ we have $\int_S|f(r, P)|^\lambda dV\leqq C^\lambda$, then

$$\int_S\sup_{0\leqq r<1}|f(r, P)|^\lambda dV\leqq\alpha C^\lambda,$$

where α depends only on λ and n. If further $\lambda\geqq 1$, there exists a function $f(P)$ of class L^λ for which

$$\lim_{r\rightarrow 1}\int_S|f(r, P)-f(P)|^\lambda dV=0.$$

This generalizes well-known results by Hardy and Littlewood and by F. and M. Riesz. The author announces further a generalization of the ergodic theorem of Birkhoff to compact Lie groups in analogy with an earlier generalization by N. Wiener [Duke Math. J. **5**, 1–18 (1939)] of the same theorem to Abelian groups. *H. Tornehave.*

Calculus of Variations

McFarlan, L. H. Special forms of the Euler differential equations when x is absent from the integrand. Univ. Washington Publ. Math. **2**, no. 3, 33–36 (1940).

For a minimizing arc $y=y(x)$ of a simple integral non-parametric variational problem with integrand function $f(y, p)$ not involving the independent variable x the well-known first integral condition $f(y, y')-y'f_p(y, y')=\text{constant}$ is derived as the transversality condition for an associated variational problem of Lagrange type. The paper contains related remarks for problems with integrand functions $f(x, y, y', \cdots, y^{(n)})$ independent of x or $y, y', \cdots, y^{(q)}$, $0\leqq q<n$. *W. T. Reid* (Evanston, Ill.).

Cinquini, S. Sopra l'esistenza dell'estremo in campi illimitati. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 675–682 (1948).

This paper presents two theorems on the existence of an absolute minimum for an integral of the calculus of variations of the form $\int_{C[n]} f(x, y(x), y'(x), \cdots, y^{(n)}(x))dx$; the established results are supplementary to earlier results of the author [Ann. Scuola Norm. Super. Pisa (2) **5**, 169–190 (1936); **6**, 191–209 (1937)]. Given examples provide a brief comparative study of the class of problems covered by a particular theorem of the second paper cited and the class of problems covered by the theorems of the present paper. *W. T. Reid* (Evanston, Ill.).

Stampacchia, G. Sulla semicontinuità degli integrali doppi, in forma ordinaria, nel calcolo delle variazioni. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 247–253 (1947).

Let $\mathfrak{M}$ denote a class of functions $u(x, y)$ continuous on a bounded open region D, and suppose that an operator $\mathfrak{D}u(x, y)$ and a system of neighborhoods $i_\rho[u]$, $\rho>0$, $u\epsilon\mathfrak{M}$, satisfy the following conditions: $\mathfrak{D}u(x, y)$ is on $\mathfrak{M}$ to the class of functions Lebesgue integrable on D; with respect to the system of neighborhoods the integral of $\mathfrak{D}u(x, y)$ over an arbitrary rectangle interior to D is continuous on $\mathfrak{M}$; if $u(x, y)$ and $u_0(x, y)$ are functions of $\mathfrak{M}$ and $u\epsilon i_\rho[u_0]$ then $|u(x, y)-u_0(x, y)|<\rho$ on D. Semi-continuity properties of the integral $\iint_D f(x, y, u(x, y), \mathfrak{D}u(x, y))dxdy$ with respect to the given system of neighborhoods are established under the following hypotheses on $f(x, y, u, w)$ for (x, y) in D and (u, w) arbitrary finite values: f is continuous and has a continuous partial derivative f_w; as a function of w, f satisfies a condition of quasi-regular positive semi-normality; f is bounded below. The method of proof is a direct extension of that used by Tonelli [Ann. Scuola Norm. Super. Pisa (2) **3**, 401–450 (1933), pp. 405–410] for simple integral nonparametric problems. The extension of these results to integrals involving more than one such operator $\mathfrak{D}u(x, y)$ is discussed briefly. *W. T. Reid* (Evanston, Ill.).

Theory of Probability

Franckx, E. L'évolution des collectivités. Assoc. Actuair. Belges. Bull. no. **51**, 31–40 (1946).

The paper also appeared in Mitt. Verein. Schweiz. Versich.-Math. **45**, 279–288 (1945); these Rev. **7**, 311.

Varoli, Giuseppe. Alcune probabilità relative al trinomio ax^2+bx+c. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **7**, 72–90 (1948).

L'auteur résoud un certain nombre de problèmes du type suivant: a, b, c étant des variables aléatoires de loi uniforme sur l'intervalle $(-\lambda, \lambda)$, quelle est la probabilité pour que le trinôme ax^2+bx+c soit positif pour tout l'intervalle $(-p, p)$; les calculs, élémentaires, sont exposés en detail. *R. Fortet* (Caen).

Mack, C. An exact formula for $Q_k(n)$, the probable number of k-aggregates in a random distribution of n points. Philos. Mag. (7) **39**, 778–790 (1948).

Suppose that n points of the interval $(0, 1)$ are picked independently and at random so that the coordinate of each point has a uniform distribution. If exactly k of the points lie within an interval of length δ they are said to form a k-aggregate. Silberstein [same Mag. (7) **36**, 319–336 (1945); these Rev. **7**, 310] found approximations to the expected number of such aggregates. The author shows that it is easy to obtain an exact expression. His method is elementary and applies also to two dimensions, although there simple expressions can be derived only if the boundary effects are neglected. *W. Feller* (Ithaca, N. Y.).

Westcott, C. H. A study of expected loss rates in the counting of particles from pulsed sources. Proc. Roy. Soc. London. Ser. A. **194**, 508–526 (1948).

The author considers counting apparatus of the familiar type with finite resolving time, but assumes that the incoming pulses have finite duration. He finds approximations to the average of the expected loss in counting, valid under various assumptions concerning the relative order of magnitude of the resolving time, the frequency of pulses, and their duration. *W. Feller* (Ithaca, N. Y.).

Feller, W. The fundamental limit theorems in probability. Revista Mat. Hisp.-Amer. (4) **8**, 95–132 (1948). (Spanish)

Translated from Bull. Amer. Math. Soc. **51**, 800–832 (1945); these Rev. **7**, 128.

Sapogov, N. A. On sums of dependent random variables. Doklady Akad. Nauk SSSR (N.S.) **63**, 353–356 (1948). (Russian)

The author finds upper limits to the error in Bernstein's version of the central limit theorem for dependent random variables [Math. Ann. **97**, 1–59 (1926)]. *J. L. Doob.*

Hollingsworth, C. A. The average boundaries of statistical chains. J. Chem. Phys. **16**, 544–547 (1948).

The author considers a chain in 3-space, whose links have the same length but are randomly and independently distributed in direction. He finds approximations (large number of links) for the maximum distance from the first link to the rest of the chain by solving the corresponding diffusion problem, which involves the usual partial differential (heat) equation. *J. L. Doob* (Urbana, Ill.).

Hollingsworth, C. A. The transverse boundary of the random coil. J. Chem. Phys. **17**, 97–99 (1949).

Using the same methods as in the paper reviewed above the author finds an asymptotic expression for the distribution of the maximum distance of any point of a random chain from the line joining its ends. *J. L. Doob.*

Fréchet, Maurice. Les éléments aléatoires de nature quelconque dans un espace distancié. Ann. Inst. H. Poincaré 10, 215–310 (1948).

The author gives a detailed treatment of the theory of random variables with values in metric spaces. Particular attention is paid to concepts of typical values, convergence, and continuity. *J. L. Doob* (Urbana, Ill.).

Harris, T. E. Branching processes. Ann. Math. Statistics 19, 474–494 (1948).

Let $z_0=1$, let z_1 be a random variable taking on only non-negative integral values, and let the conditional distribution of z_{n+1} for given $z_0, \cdots, z_n$ be that of the sum of z_n random variables each having the distribution of z_1. Then z_n can be interpreted as the number of individuals in the nth generation of a certain type of birth process. It is supposed that the expectations $E\{z_1\}, E\{z_1^2\}$ are finite. Suppose that there is positive probability that z_n will remain positive for all n. Then it is shown that $\lim_{n\to\infty} z_n/E\{z_n\}=w$ exists. [It follows from a theorem of the reviewer, Trans. Amer. Math. Soc. 47, 455–486 (1940), theorem 1.3; these Rev. 1, 343, that this limit in the mean exists as a limit with probability 1 if $E\{z_1\}<\infty$.] The distribution function $G(u)$ of w is shown to be absolutely continuous, with a continuous derivative, for $u>0$, and results are obtained on the behavior of $G(u)$ for $u\to 0$ and $u\to\infty$. Various special cases are worked out in detail. Maximum likelihood estimates of $\Pr\{z_1=r\}$ and of $E\{z_1\}$ are derived. If there is probability 1 that $z_n=0$ for large n, a method is given for obtaining the moment generating function of the smallest n for which $z_n=0$.
J. L. Doob (Urbana, Ill.).

Bellman, Richard, and Harris, Theodore E. On the theory of age-dependent stochastic branching processes. Proc. Nat. Acad. Sci. U. S. A. 34, 601–604 (1948).

The authors consider a branching stochastic process in which any particle coming into existence at time s has a certain probability of being transformed into $n\geqq 1$ particles by time $s+t$; the time t and the number n are both random variables, with distributions independent of s. Let $Z(t)$ be the number of particles existing at time t, and suppose that $Z(0)=1$. The following results are announced, generalizing work done by Yaglom [Doklady Akad. Nauk SSSR (N.S.) 56, 795–798 (1947); these Rev. 9, 149] and Harris [see the preceding review] in the case when each particle has a fixed lifetime. The generating function of $Z(t)$ satisfies an explicitly given nonlinear integral equation, which determines it uniquely. The expectation $E\{Z(t)\}$ satisfies the linear integral equation of renewal theory; the character of the moments of $Z(t)$ as $t\to\infty$ is then deduced from Feller's work on this equation [Ann. Math. Statistics 12, 243–267 (1941); these Rev. 3, 151]. The random variable $Z(t)/E\{x(t)\}$ converges in probability to a random variable which, under restrictive hypotheses on the lifetime of a particle, has a density of distribution. *J. L. Doob* (Urbana, Ill.).

Wold, Herman O. A. On prediction in stationary time series. Ann. Math. Statistics 19, 558–567 (1948).

Stationary sequences of random variables (stationary stochastic processes) have been studied in detail in recent years. The author points out that, if for any sequence of constants $\cdots, c_0, c_1, \cdots$ we denote $\lim (t_2-t_1+1)^{-1}\sum_{t_1}^{t_2} c_t$ ($t_1\to-\infty, t_2\to+\infty$) by $M\{c_t\}$, then if a sequence x_t has the property that $M\{x_t\}$ and $M\{x_t x_{t+k}\}$ exist (all k) then the latter mean has the attributes of a covariance function and the theorems on stationary processes can be interpreted as theorems on the individual sequence $\{x_t\}$ and its translations. He gives the details for linear least squares prediction. Conversely almost all sample sequences of a stationary process satisfy the conditions imposed on x_t above, so that the analysis of a process has as its counterpart that of the sample sequences. *J. L. Doob* (Urbana, Ill.).

Ionescu Tulcea, et Marinescu, G. Sur certaines chaînes à liaisons complètes. C. R. Acad. Sci. Paris 227, 667–669 (1948).

Let $P(t, A)$ be a function defined for t in a compact metric space and A a set of a Borel field of sets of a space E. For fixed t, $P(t, A)$ is completely additive in A, and $0\leqq P(t, A)\leqq P(t, E)\leqq 1$. For each $x\epsilon E$, $y(t, x)$ is a function with range in t space. Both $P(t, A)$ and $y(t, x)$ have uniformly bounded difference quotients (bound less than 1) in t. Then the operator defined by $Tf=\int_E f[y(t, x)]P(t, dx)$ is a quasi-completely continuous operator on a certain Banach space of functions. The norms of the powers of T are uniformly bounded and 1 is a characteristic value if $P(t, E)=1$. In the latter case the function $P(t, A)$ can be considered as a stochastic transition function, where t represents the past values assumed by a system, and A is the set into which the system is to go at the next transition. The known properties of the iterates of quasi-completely continuous operators then imply properties of the iterates of the given transition probabilities. This method was used, for t spaces with a finite number of points, by Doeblin and Fortet [Bull. Soc. Math. France 65, 132–148 (1937)]. In particular, if the process is a Markov process, t space is E, and in this case the method was used, with less restrictive hypotheses, by Yosida and Kakutani [Ann. of Math. (2) 42, 188–228 (1941); these Rev. 2, 230]. *J. L. Doob* (Urbana, Ill.).

Onicescu, Octav, et Mihoc, Gh. Les chaînes de variables aléatoires. Problèmes asymptotiques. Acad. Roum. Études Recherches 14, 167 pp. (1943).

The authors give a treatment of the asymptotic theory of Markov chains, with some attention to the more general chains "à liaisons complètes," using the methods (usually involving characteristic functions) developed by them in a series of earlier papers. *J. L. Doob* (Urbana, Ill.).

Baptist, J.-H. Étude de la dépendance stochastique. Assoc. Actuair. Belges. Bull. no. 50, 15–36 (1945).

Expository paper. *J. L. Doob* (Urbana, Ill.).

Burgers, J. M. Spectral analysis of an irregular function. Nederl. Akad. Wetensch., Proc. 51, 1073–1076 (1948).

The author studies the effect of a filter on a stationary type signal. He is evidently unfamiliar with modern harmonic analysis. *J. L. Doob* (Urbana, Ill.).

Middleton, David. Spurious signals caused by noise in triggered circuits. J. Appl. Phys. 19, 817–830 (1948).

L'auteur envisage le problème suivant: étant donnée une fonction aléatoire $V(t)$ d'une variable réelle t, quelle est l'espérance mathématique $n(t)$ du nombre de fois que, dans l'intervalle $(0, t)$, $V(t)$ franchit en croissant une valeur déterminée quelconque V_0? Sous certaines conditions [que l'auteur n'a pas cherché à établir], ce problème a un sens et $n(t)$ est donné par la formule:

$$n(t)=\int_0^t\left\{\int_0^\infty u\rho(V_0, u; \tau)du\right\}d\tau,$$

où $\rho(v, u; \tau)dvdu$ est la probabilité élémentaire pour que:

$v < V(\tau) < v+dv$, $u < dV(\tau)/d\tau < u+du$; l'auteur applique ce résultat au cas où $V(t)$ est de la forme: $V(t)=V_1(t)+V_2(t)$, où $V_1(t)$ est une fonction certaine (signal) et $V_2(t)$ une fonction aléatoire stationnaire laplacienne (bruit de fond); il donne des expressions explicites de $n(t)$ pour divers types de $V_1(t)$ et de $V_2(t)$ (ceux-ci caractérisés par leur fonction spectrale) intéressants pour les applications au problème des "circuits à amorçage." L'auteur rappelle que le cas $V_1(t)=0$, $V_0=0$ a déjà été traité par S. O. Rice [Amer. J. Math. **61**, 409–416 (1939)] et M. Kac [Bull. Amer. Math. Soc. **49**, 314–320 (1943); ces Rev. **4**, 196]. *R. Fortet* (Caen).

Mathematical Statistics

Banerjee, D. P. On the cumulants of β_2. Bull. Calcutta Math. Soc. **40**, 76 (1948).

The author attempts to find the expected value of $b_2=m_4/m_2^4$ from a normal parent. The result is incorrect.
L. A. Aroian (New York, N. Y.).

Risser, R. Note relative aux surfaces de probabilités. Assoc. Actuair. Belges. Bull. no. **53**, 5–48 (1948).

The author is interested in generalizations of the univariate Pearson system to higher-dimensional distributions. The main results are as follows. (1) Approximation of a two-dimensional multinomial and a two-dimensional hypergeometric probability function by a two-variable Edgeworth series and a two-variable fourth degree exponential probability function. (2) The solutions of the partial differential equation $z^{-1}z_{xy}=H(x, y)$, which may be of value as probability surfaces. (3) The solutions of the system $\partial \log z/\partial x=f(x, y)/\varphi(x, y)$, $\partial \log z/\partial y=g(x, y)/\psi(x, y)$, including the hyperbolic paraboloid and the elliptic paraboloid as probability surfaces, the generalizations of Laplace's first type and of the Pearson univariate system.

A considerable portion of (3) has already been developed by van Uven [Nederl. Akad. Wetensch., Proc. **50**, 1063–1070, 1252–1264 (1947); **51**, 41–52, 191–196 (1948) = Indagationes Math. **9**, 477–484, 578–590 (1947); **10**, 12–23, 62–67 (1948); these Rev. **9**, 363, 452] in a form readily adaptable for the graduation of observed bivariate distributions. *L. A. Aroian* (New York, N. Y.).

Aitken, A. C. On a problem in correlated errors. Proc. Roy. Soc. Edinburgh. Sect. A. **62**, 273–277 (1948).

Consider n independent pairs $(x_i+\xi_i, y_i+\eta_i)$ of observations, where ξ_i and η_i are errors normally distributed with variances σ_1^2, σ_2^2 and product moment $\rho\sigma_1\sigma_2$. Let x and y be represented by

$$x=a_0+a_1p_1(t)+\cdots+a_{k-1}p_{k-1}(t),$$
$$y=b_0+b_1p_1(t)+\cdots+b_{k-1}p_{k-1}(t),$$

where the $p_j(t)$ are linearly independent basic functions. The author shows that the mean value of $\sum(x_i-\bar{x})(y_i-\bar{y})$ is $(n-k)\rho\sigma_1\sigma_2$ and the sampling variance is $(n-k)(1+\rho^2)\sigma_1^2\sigma_2^2$. He also discusses the case where x and y are represented by different sets of basic functions instead of by the same set as shown above. The case of more than two variables, $(x, y, z, \cdots)$, is briefly examined. *W. E. Milne.*

Steinhaus, H. Sur l'interprétation des résultats statistiques. Colloquium Math. **1**, 232–238 (1948).

Expository paper. The author bases himself on the assumption of a uniform a priori distribution of the parameter to be estimated. *J. Wolfowitz* (New York, N. Y.).

Fisher, R. A. Conclusions fiduciaires. Ann. Inst. H. Poincaré **10**, 191–213 (1948).

A brief expository account of the author's well-known statistical methods. The statistical estimation of parameters in a distribution is introduced on a quasi-philosophical basis as an example of the logic of induction and is regarded as a mathematical counterpart of deduction, familiar in other mathematical theories. The phrase "fiduciary probability" is used to distinguish the degree of belief with which such inductions are made from ordinary probability, which would seem to require Bayes' theorem. Some examples are given which illustrate precision, and quantity of information, and similar concepts in the author's system. The paper contains nothing essentially new from the scientific point of view. The situation is left as devoid of axiomatization of fiduciary probability and its relationship with ordinary probability as ever. *B. O. Koopman* (New York, N. Y.).

Anderson, T. W. On the theory of testing serial correlation. Skand. Aktuarietidskr. **31**, 88–116 (1948).

Let the joint density function of n chance variables at the point $x_1, \cdots, x_n$ be

$$K\cdot\exp\{-\tfrac{1}{2}\alpha(x-\mu)'\psi(x-\mu)+\lambda(x-\mu)'\theta(x-\mu)\},$$

where K, λ and $\alpha>0$ are scalar constants, ψ and θ are $n\times n$ matrices, of which ψ is positive definite and θ symmetric. The constant λ is restricted to values for which $\psi+\lambda\theta$ is positive definite, x is the column vector with elements $x_1, \cdots, x_n$, and μ the column vector of expected values. It is given that $\mu=\sum_1^m\beta_\nu\varphi_\nu$, where the β's are scalars and the φ's are $m(<n)$ given, linearly independent, vectors. The quantities $\alpha, \lambda, \beta_1, \cdots, \beta_m$ are unknown (the constant K is of no consequence in the ensuing). The statistical problem is to test the hypothesis $H_0:\lambda=0$, the size of type I error being fixed at ϵ. A precise statement of the author's results would require too much space, but the gist is the following. (1) All similar critical regions for testing H_0 are such that a constant fraction ϵ of the area of each member of a certain family of manifolds in n-space is in the critical region. (2) A test based on the quotient of two quadratic forms is most powerful for alternatives $\lambda>0$ (or $\lambda<0$) if the vectors φ are linear combinations of m of the characteristic vectors of $\theta-\omega\psi$. (3) Under the latter circumstances the type B_1 test exists. Suitable specialization of the matrices ψ and θ yields results about various problems involving serial correlation. A detailed discussion of these is given.

The reviewer notes the following errors. The matrix θ should be postulated to be symmetric. The factor ϵ has been omitted from the right member of (5). In (5) the integration should be with respect to the element of area on the manifolds involved. On page 94, line 10 from the bottom, the inequality $\lambda>0$ should be reversed. The symbol K is used throughout to represent different constants.
J. Wolfowitz (New York, N. Y.).

Pólya, George. Exact formulas in the sequential analysis of attributes. Univ. California Publ. Math. (N.S.) **1**, 229–239 (1948).

A method for obtaining exact formulas for the sequential probability ratio test is given in the case when the distribution is binomial and the common slope of the acceptance and rejection lines is rational. [Another method for obtaining exact formulas in a more general class of cases was given by the reviewer [Ann. Math. Statistics **15**, 283–296 (1944); these Rev. **6**, 88] and by M. A. Girshick, using a different method of approach [Ann. Math. Statistics **17**, 282–298

(1946); these Rev. **8**, 163]. The paper under review was submitted for publication before Girshick's paper appeared and presents a method of attack which is of interest in itself.] *A. Wald* (New York, N. Y.).

Robinson, Julia. A note on exact sequential analysis. Univ. California Publ. Math. (N.S.) **1**, 241–246 (1948).

An explicit formula is given for the special case when the common slope of the acceptance and rejection lines is the reciprocal of an integer. Also some numerical results are presented which were computed by Pólya's method [see the preceding review]. These numerical computations are intended to throw some light on the following two questions. (1) What is the size of the error in the approximation formulas given in the literature? (2) Does there exist a test procedure which gives better results than the sequential probability ratio test? Lower and upper bounds for the error given by the reviewer [Ann. Math. Statistics **16**, 117–186 (1945); these Rev. **7**, 131] show that the error is small when the excess of the cumulative sum over the boundaries at the termination of the test is small, but may be substantial when the excess is considerable. The numerical findings in the paper under review seem to be in agreement with this. The comparisons made by the author do not answer the second question conclusively. It has been answered in the negative sense by the reviewer and J. Wolfowitz in a paper [Ann. Math. Statistics **19**, 326–339 (1948); these Rev. **10**, 201] which appeared after the publication of the note under review. *A. Wald.*

Plackett, R. L. Boundaries of minimum size in binomial sampling. Ann. Math. Statistics **19**, 575–580 (1948).

In sequential sampling from a binomial population, sampling is terminated when a point with coordinates x (successes) and y (failures) reaches a boundary B. If $n=\max(x+y)$ on B then B has at least $n+1$ points. Minimum B's have exactly $n+1$ points and in this case $N(x, y)$, the number of paths to (x, y) on B, is determined by equating coefficients of p in $\sum_B N(x, y)p^x(1-p)^y \equiv 1$. Accessible points in a region bounded by a minimum B (and the axes) form a simple region (convex relative to lines $x+y=a$). [See Girshick, Mosteller, and Savage, same Ann. **17**, 13–23 (1946); these Rev. **8**, 477.] *A. M. Mood.*

Kempthorne, O. A simple approach to confounding and fractional replication in factorial experiments. Biometrika **34**, 255–272 (1947).

The author observes that confounded designs may be considered as designs with fractional replication if the blocks are considered as factors. Various designs with fractional replication are considered from this point of view. The multifactorial designs of Plackett and Burman [Biometrika **33**, 305–325 (1946); these Rev. **8**, 44] may be considered as high order factorial designs with fractional replication. *H. B. Mann* (Columbus, Ohio).

Johnson, N. L. Alternative systems in the analysis of variance. Biometrika **35**, 80–87 (1948).

The author discusses, for the case of one and two way designs, a set-up alternative to the customary linear hypothesis in which some of the parameters of the linear hypothesis are considered to be random variables. Situations where this set-up is appropriate arise often in practice, for instance, when randomly selected samples of manufactured articles are taken to test for homogeneity in the material. In the case of a one way design the critical region arrived at from the linear hypothesis is still correct but the power function becomes different and easier to compute. In the case of two way designs with several observations in each subclass, if the interaction is considered as a random variable and the main effects as constants, the critical region for testing the main effects corresponding to a linear hypothesis which makes no assumption about the interactions becomes too large. The author also discusses randomized designs in their relation to the two set-ups discussed in his paper. *H. B. Mann* (Columbus, Ohio).

Brownlee, K. A., Kelly, B. K., and Loraine, P. K. Fractional replication arrangements for factorial experiments with factors at two levels. Biometrika **35**, 268–276 (1948).

It is desired to construct fractional replications in which main effects and first order interactions have aliases which are interactions of order 2 or higher. The subgroup of aliases of unity thus must apart from unity contain only elements involving 5 or more letters. This leads to the general problem of enumerating in a group of order p^n and type $(1, \cdots, 1)$ all subgroups such that every element except unity in them involves at least m letters. The authors give a method of tactical enumeration for the case $p=2$ and apply it to construct the arrangements in the title for 16, 32 and 64 plots. Some arrangements involving factors at four levels are also constructed. *H. B. Mann.*

Brownlee, K. A., and Loraine, P. K. The relationship between finite groups and completely orthogonal squares, cubes and hyper-cubes. Biometrika **35**, 277–282 (1948).

Factorial designs of factors on p levels, where p is a prime, designed in sets of orthogonal Latin squares constructed by the customary method from the residue system modulo p, may be regarded as fractional replications. In all these designs the main effects then have aliases which are first order interactions. Similarly, in a set of orthogonal Latin hypercubes in n dimensions constructed in the customary way the main effects have aliases which are $(n-1)$th order interactions. *H. B. Mann* (Columbus, Ohio).

Banerjee, K. S. On the design of experiments for weighing and making other types of measurements. Science and Culture **13**, 344 (1948).

The author remarks that the weighing designs proposed by K. Kishen [Ann. Math. Statistics **16**, 294–300 (1945); these Rev. **7**, 133] may be improved by adding instead of rows $11\cdots 1$ other rows. As an example he gives a design for the weighing of 3 objects in 6 weighings which has smaller variance than that proposed by Kishen.

H. B. Mann (Columbus, Ohio).

Mathematical Economics

***Malmquist, Sten. A Statistical Analysis of the Demand for Liquor in Sweden. A Study of the Demand for a Rationed Commodity.** Thesis, University of Uppsala, 1948. 135 pp.

The statistical methods used are those of classical regression analysis, and do not take account of recent development in the fitting of simultaneous economic relations. Principal

interest attaches to the theoretical development of the aggregation of individual price- and income-elasticities of demand into market elasticities, especially in the case where the commodity is sold under a rationing scheme. The treatment is casuistic and does not permit of easy summary; the dependence of the results on distributional assumptions is clearly shown. Similar results are obtained for the elasticity of demand with respect to the size of the ration. All results are illustrated by fits to actual data. *K. J. Arrow.*

Nataf, André. Sur la possibilité de construction de certains macromodèles. Econometrica 16, 232–244 (1948).

There is proposed a solution to an aggregation problem raised by Klein [Econometrica 14, 93–108 (1946)]. Assume individual production functions

$$(1)\qquad F_\alpha(x_{1\alpha}, \cdots, x_{m\alpha}; n_{1\alpha}, \cdots, n_{r\alpha}; z_{1\alpha}, \cdots, z_{s\alpha})=0, \quad \alpha=1, 2, \cdots, A,$$

which state that the αth firm produces m commodities $\{x_{i\alpha}\}$ by means of r kinds of labor service $\{n_{j\alpha}\}$ and s kinds of capital service $\{z_{k\alpha}\}$. It is further assumed that (1) may be written as

$$(2)\qquad x_{1\alpha}=f_\alpha(x_{2\alpha}, \cdots, x_{m\alpha}; n_{1\alpha}, \cdots, n_{r\alpha}; z_{1\alpha}, \cdots, z_{s\alpha}), \quad \alpha=1, 2, \cdots, A.$$

The relevant aggregates are defined by the transformation functions

$$(3)\qquad \begin{aligned} X&=G(x_{i\alpha}), & i&=1, 2, \cdots, m;\ \alpha=1, 2, \cdots, A,\\ N&=H(n_{j\alpha}), & j&=1, 2, \cdots, r;\ \alpha=1, 2, \cdots, A,\\ Z&=I(z_{k\alpha}), & k&=1, 2, \cdots, s;\ \alpha=1, 2, \cdots, A. \end{aligned}$$

The transformed variables are functionally related by a relation (4) $\phi(X, N, Z)=0$ if the transformation matrix

$$(5)\qquad \left\|\begin{matrix} \left[\dfrac{\partial G}{\partial x_{1\alpha}}\dfrac{\partial x_{1\alpha}}{\partial x_{i\alpha}}+\dfrac{\partial G}{\partial x_{i\alpha}}\right] & 0 & 0 \\ \left[\dfrac{\partial G}{\partial x_{1\alpha}}\dfrac{\partial x_{1\alpha}}{\partial n_{i\alpha}}\right] & \left[\dfrac{\partial H}{\partial n_{i\alpha}}\right] & 0 \\ \left[\dfrac{\partial G}{\partial x_{1\alpha}}\dfrac{\partial x_{1\alpha}}{\partial z_{i\alpha}}\right] & 0 & \left[\dfrac{\partial I}{\partial z_{i\alpha}}\right] \end{matrix}\right\|,$$

where the elements are column vectors, is of rank 2. The aggregative functions (3) must satisfy the conditions that all 3-rowed determinants vanish identically, where the $\partial H/\partial n_{i\alpha}$ and $\partial I/\partial z_{i\alpha}$ are not all zero.

The author shows that in order to specify a relation (4) the individual production functions (1) must assume the form

$$\theta_\alpha(x_{1\alpha}, \cdots, x_{m\alpha})=\chi_\alpha[H_\alpha(n_{1\alpha}, \cdots, n_{r\alpha})+I_\alpha(z_{1\alpha}, \cdots, z_{s\alpha})], \quad \alpha=1, 2, \cdots, A.$$

The aggregates must then be defined in the form

$$H\equiv\sum_{\alpha=1}^{A} H_\alpha, \quad I\equiv\sum_{\alpha=1}^{A} I_\alpha, \quad G\equiv\sum_{\alpha=1}^{A} \chi_\alpha^{-1}(\theta_\alpha),$$

and hence $G\equiv I+H$, $X=N+Z$. *M. P. Stoltz.*

Giaccardi, F. Alcune considerazioni sulle "curve dei redditi" di Amoroso e di Gibrat. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 81–82, 67–74 (1948).

Let $F(x)dx$ be the number of persons whose incomes lie between x and $x+dx$. Define $\epsilon=d\log F(x)/d\log x$. Let $A>0$ and $c\geqq 0$ be two constants. It is shown that the differential equation $d\epsilon/dx=-Ax^{-c}$ defines a class of functions $F(x)$ which contains the following functions variously suggested in the past to describe the observed distribution of incomes: for $c=0$, $F(x)=Cx^He^{-\gamma x}$ [Pearson's type III]; for $c=1$, $F(x)=Cx^He^{-\gamma(\log x)^2}$ [McAlister and Gibrat]; for $c=2$, $F(x)=Cx^He^{-\gamma/x}$ [Pearson's type V]; for $c=+\infty$, $F(x)=Cx^H$ [Pareto's first approximation]; in general, for $c\geqq 0$, $c\neq 1$, $F(x)=Cx^He^{-\gamma x^{1-c}}$ [Amoroso], where C, γ, H are constants, and H is a limit of ϵ. *J. Marschak* (Chicago, Ill.).

Mathematical Biology

✶Malécot, G. Les Mathématiques de l'Hérédité. Masson et Cie., Paris, 1948. viii+65 pp. 180 francs.

In the first part the author discusses the elementary consequences of the Mendelian theory with particular reference to the correlation among related individuals and to finite populations. In the second part the theory of gene evolution is discussed. In the main the author follows S. Wright. He derives the differential equation for the distribution of gene frequencies in a somewhat more complicated way than was done originally by Wright [Proc. Nat. Acad. Sci. U. S. A. 31, 382–389 (1945); these Rev. 7, 319]. The underlying assumptions are stated only in part. Actually the differential equation in question is of the Fokker-Planck type and its validity (under much more general conditions) is a direct consequence of diffusion theory. [This remark, due to Kolmogorov, is made in Wright's paper quoted above.] *W. Feller* (Ithaca, N. Y.).

Malécot, Gustave. Le regroupement des classes d'une table de contingence et ses applications à la génétique. C. R. Acad. Sci. Paris 226, 1682–1683 (1948).

The regrouping scheme mentioned in the title is as follows. In an infinite population the elements are classified by a double dichotomy so that each element belongs to one of the classes A_i and, at the same time, to a class B_j. The elements are then reclassified into classes C_m and D_n in such a way that the probabilities of an element's belonging to C_m and D_n depend, respectively, only on its class A_i or B_j. It is shown that this scheme applies in genetics if the population consists of parental couples or their descendants and the classes are, respectively, the genotypes of the male and female parents. *W. Feller* (Ithaca, N. Y.).

De Donder, Th. Le calcul des variations introduit dans l'étude des espèces et des variétés. IV. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 34, 229–231 (1948).

The concepts developed by the author in previous notes [same Bull. Cl. Sci. (5) 33, 502–506, 718–724 (1947); 34, 122–125 (1948); these Rev. 9, 297, 604] are interpreted in terms of mutation and speciation. *A. S. Householder.*

Rapoport, Anatol, and Shimbel, Alfonso. Steady states in random nets. I. Bull. Math. Biophys. 10, 211–220 (1948).

A semi-infinite linear array of neurons is considered in which $K(x, \xi)d\xi$ is the "probability that a neuron at x

receives an axone from a neuron lying in the interval $(\xi, \xi+d\xi)$." Assuming a "firing pattern" $\phi(x)$ (i.e., a distribution of stimuli) applied and repeated at every instant, the authors investigate the possible steady states of activity. In the particular case $K(x, \xi)=k \exp[-k(x-\xi)]$, $x\geqq\xi$; $K(x, \xi)=0$, $x<\xi$, the equation can be solved in terms of quadratures for output from a given input, or input for a required output.

A. S. Householder (Oak Ridge, Tenn.).

Rapoport, Anatol. Steady states in random nets. II. Bull. Math. Biophys. 10, 221–226 (1948).

[Cf. the preceding review.] The author's abstract is as follows. Two semi-infinite chains are considered interacting as random nets. Conditions for steady state are derived for the cases of cross-excitatory and cross-inhibitory association connections. In the cross-inhibitory case a unique non-trivial self-reproducing steady state is shown to exist.

A. S. Householder (Oak Ridge, Tenn.).

TOPOLOGY

Ratib, Ismail. Une proposition sur les régions que présentent certains réseaux cubiques. Proc. Math. Phys. Soc. Egypt **3** (1947), 53–57 (1948).

The author considers the case of a connected graph having no isthmus, in which each edge is incident with just two nodes and each node with just three edges, and which is imbedded in a sphere. He shows that if the resulting map has more than four regions, and if some region has a side in common with each of the others, then there are two regions each of less than four sides which have no side in common.

W. T. Tutte (Toronto, Ont.).

Gau, P.-E. A propos d'une formule d'Euler. Bull. Sci. Math. (2) **72**, 36–39 (1948).

This is an expository paper showing that the faces of any simply-connected polyhedron include at least one triangle, quadrangle or pentagon. [See Catalan, J. École Polytech. **24**, cahier **41**, 1–71 (1865), in particular, p. 5.]

H. S. M. Coxeter (New York, N. Y.).

Borsuk, K. Concerning the Euler characteristic of normal spaces. Colloquium Math. **1**, 206–209 (1948).

Let the normal space A be expressed as a union of a finite number of closed sets, $A_1, \cdots, A_k$. Let any nonvoid intersection of these sets be acyclic, i.e., have the same homology groups, with rational coefficients, as a point. Then the Euler characteristic $\chi(A)$ of A exists and is equal to $\chi(N)$, where N is the nerve of the collection $A_1, \cdots, A_k$. The proof makes use of Čech's extension to normal spaces of the Mayer-Vietoris formula, but except for this it is quite elementary.

E. G. Begle (New Haven, Conn.).

Borsuk, Karol. Sur un espace compact localement contractile qui n'est pas un rétracte absolu de voisinage. Fund. Math. **35**, 175–180 (1948).

The author proved [C. R. Acad. Sci. Paris **194**, 951–953 (1932)] that every finite-dimensional compact metric space which is locally contractible is an absolute neighborhood retract. The infinite dimensional case has remained as an open problem. The author now shows that the solution is negative. Indeed, he constructs a very simple closed subset of the Hilbert cube which is locally contractible and which can be retracted on subsets homeomorphic with the n-sphere for any $n>0$. Taking the join of this space with a single point, an example is obtained of a compact metric contractible and locally contractible space which is not an absolute retract.

S. Eilenberg (New York, N. Y.).

Tong, Hing. On some problems of Čech. Ann. of Math. (2) **50**, 154–157 (1949).

By means of examples, the author answers questions posed by E. Čech in the conclusion of the latter's article on embedding [Ann. of Math. (2) **38**, 823–844 (1937)]. The answers follow. (1) There are completely regular spaces S such that $\beta(S)-S$ consists of one point. (2) There are completely regular spaces in which no open set has a normal closure. (3) There are completely (=hereditarily) locally normal spaces which are not locally completely normal. (4) There are locally completely normal spaces which are not normal. While the space exhibited for (2) is somewhat more complex, those used for the others are subspaces of $C\times D$ where C is the closed interval of ordinals $[1, \omega]$ and D the closed interval $[1, \omega_1]$, where ω_1 is the first uncountable ordinal. Problem (1) is the simplest of the four, but it might be pointed out that its solution is also obvious from a lemma [E. Hewitt, Ann. of Math. (2) **47**, 503–509 (1946); these Rev. **8**, 165] which in its simplest form yields the space and argument used in the present article.

R. Arens (Los Angeles, Calif.).

Smirnov, Yu. On the theory of completely regular spaces. Doklady Akad. Nauk SSSR (N.S.) **62**, 749–752 (1948). (Russian)

Let R be a T_1-space. A system Σ of pairs $(\phi, O\phi)$, ϕ closed, $O\phi\supset\phi$ open, is called dense if, for any $(\phi, O\phi)\epsilon\Sigma$, there is a neighborhood $O'\phi$ of ϕ such that $O\phi\supset[O'\phi]$, where $[O'\phi]$ denotes the closure of $O'\phi$, and $([O'\phi], O\phi)\epsilon\Sigma$. The maximal dense system is called the system of regularity of R. The author states the following theorems. (1) Disjoint closed sets ϕ_0, ϕ_1 are functionally separated (i.e., $f(\phi_0)=0$, $f(\phi_1)=1$ for some continuous real function f over R) if, and only if, $(\phi_0, R-\phi_1)$ is contained in the system of regularity of R. (2) A nonconstant continuous real function over R exists if, and only if, the system of regularity contains some $(\phi, O\phi)$, ϕ nonvoid, $O\phi\neq R$. Then the author considers extensions of continuous real functions and proves that a continuous real function f defined over a closed $\phi\subset R$ may be extended to a continuous real function over R different from 0 at every point $x\epsilon R-F$ if, and only if, the set $f^{-1}(O)$ is a G_δ in R. Finally, the author gives a method [cf. A. Tychonoff, Math. Ann. **102**, 544–561 (1929)] for constructing completely regular nonnormal spaces without using ordinal numbers.

M. Katětov (Prague).

Katětov, Miroslaw. Complete normality of Cartesian products. Fund. Math. **35**, 271–274 (1948).

Point set theoretic results such as the following are proved. (1) If the compact Hausdorff space P is not metrizable, then P^3 is not completely normal. It is unknown whether one can here replace P^3 by P^2. (2) If $P_1, P_2, \cdots$ are Hausdorff spaces and $P_1\times\cdots\times P_n$ is perfectly normal (i.e., normal and such that each closed set is a G_δ) for every n, then $P_1\times\cdots\times P_n\times\cdots$ is perfectly normal. Examples are given, one showing that $P\times P$ can be perfectly normal although P is not.

R. Arens (Los Angeles, Calif.).

Fox, R. H. On a problem of S. Ulam concerning Cartesian products. Fund. Math. **34**, 278–287 (1947).

Durch Konstruktion von Beispielen wird gezeigt, dass es Räume A_1 und A_2 gibt, welche nicht homöomorph sind, deren Cartesische Quadrate $A_1 \times A_1$ und $A_2 \times A_2$ aber homöomorph sind. Die Beispiele A_i sind berandete 4-dimensionale Mannigfaltigkeiten der Form $A_i' \times E$, wo E die Einheitsstrecke ist und A_i' aus dem Linsenraum $L(p, q_i)$ durch Entfernen des Innern einer 3-Zelle entsteht, mit geeigneten Zahlen p, q_1, q_2. Es wird bewiesen, dass $A_1 \neq A_2$ ($=$ bedeute hier "homöomorph") mit Hilfe der folgenden Invarianten σ drei-dimensionaler Mannigfaltigkeiten, die für den Rand von A_i berechnet wird: man ordnet den Elementen endlicher Ordnung der Fundamentalgruppe in natürlicher Weise Eigenverschlingungszahlen zu, und σ ist deren Wertevorrat. Dann wird $A_1 \times A_1 = A_2 \times A_2$ auf $A_1 \times E = A_2 \times E$ zurückgeführt und letzteres durch explizite Konstruktion nachgewiesen. Dies zeigt gleichzeitig, dass für Cartesische Produkte die Kürzungsregel (aus $A_1 \times B = A_2 \times B$ folgt $A_1 = A_2$) i.A. nicht gilt [vgl. J. H. C. Whitehead, Ann. of Math. (2) **41**, 825–832 (1940); diese Rev. **2**, 73]. Hierfür werden noch andere ganz einfache Beispiele angegeben und diskutiert. Schliesslich wird gezeigt, dass für kompakte Mannigfaltigkeiten (mit oder ohne Rand) der Dimension $\leqq 2$ aus $A_1 \times A_1 = A_2 \times A_2$ stets $A_1 = A_2$ folgt. *B. Eckmann.*

Bockstein, M. Un théorème de séparabilité pour les produits topologiques. Fund. Math. **35**, 242–246 (1948).

Let S be a set of spaces satisfying the second axiom of countability. For each $X \varepsilon S$ select a fixed, at most countable, basis $\mathfrak{X}$. Let T be the Cartesian product of the spaces in S. Using only the open sets of the various $\mathfrak{X}$ selected, introduce in T an open basis in the usual way (hyperslices, etc.). The unions of at most countably many basic open sets are termed of countable construction; by arbitrary unions, all the open sets in the product topology of T are, of course, obtained. It is here proved that if U_n ($n=1, 2$) are disjoint open sets in T, then disjoint open sets V_n of countable construction exist such that U_n is contained in V_n. This is moreover extended to the case in which n runs through all integers. Further generalization is limited because any set of pairwise disjoint open sets in T would be at most countable. *R. Arens* (Los Angeles, Calif.).

Bokštein, M. On the dimension of a topological product. Doklady Akad. Nauk SSSR (N.S.) **63**, 221–223 (1948). (Russian)

This announces a complete analysis of the dimension of a topological product $A \times B$ of bicompact spaces, based upon a knowledge of the homology-dimensions of A and B over certain designated coefficient groups. These groups, denoted by R, R_p, C_p, Q_p, are not identified in the note; references for this and other points of notation and of fact are to earlier papers by the author [C. R. (Doklady) Acad. Sci. URSS (N.S.) **37**, 243–245 (1942); **38**, 187–189 (1943); **40**, 339–342 (1943); same Doklady (N.S.) **59**, 631–633 (1948); these Rev. **5**, 48, 104; **6**, 97; **9**, 523]. The pertinence of these groups lies in the fact (stated as an older result of the author) that they give sufficient knowledge for the problem.

The author introduces new invariants of four types. The first, denoted by $D_0(A)$, is defined to be the largest integer q for which there exists a subset A' of A whose q-dimensional homology group over integral coefficients contains an element of infinite order. The set A' is required to be of the topological type of a set-difference of two open subsets of a bicompact space. The other invariants, $d_p(A)$, $\Delta_p(A)$, $\delta_p(A)$, p a prime, have analogous but somewhat more intricate definitions depending in part on certain projection spectra not defined in the paper. Formulas are given for calculating these invariants for a topological product, e.g., $D_0(A \times B) = D_0(A) + D_0(B)$. Another one is:

$$\Delta_p(A \times B) = \max \{\Delta_p(A) + \Delta_p(B),\, d_p(A) + \Delta_p(B),\, \Delta_p(A) + d_p(B),\, \delta_p(A) + \delta_p(B) + 1\}.$$

These invariants are related to the homology dimensions of A by formulas of which these two may suffice: $\dim_R A = D_0(A)$, and $\dim_{R_p} A = \max \{D_0(A), \Delta_p(A)+1\}$.

The principal theorem states that the homology dimension of $A \times B$ for coefficient groups R and C_p is the sum of the corresponding dimensions, and gives formulas for calculating this dimension for the coefficients R_p and Q_p. The formula for the group Q_p reads:

$$\dim_{Q_p} (A \times B) = \max \{\dim_{Q_p} A + \dim_{Q_p} B,\, \dim_{C_p} A + \dim_{C_p} B - 1\}.$$

The author remarks that one of his earlier papers, the third referred to above, has certain lacunae which the present work will take into account. *L. Zippin.*

Lihtenbaum, L. M. On mappings of discrete Linfield spaces. Mat. Sbornik N.S. **23(65)**, 315–328 (1948). (Russian)

This is a study of a dimension-theory for "discrete spaces" [B. Z. Linfield, Espace Discret Paramétrique et Non Paramétrique, Paris, 1925]: in such spaces certain pairs of points are designated as neighbors, the relation being reflexive and symmetric. The set of points of such a space G is called the basis of G. A space G^* is a subspace of G if the basis of G^* is a subset of the basis of G and if two points of G^* are neighbors (in G^*) only if there are neighbors in G; G^* is a principal subspace if moreover two points of G^* are neighbors in G^* whenever they are neighbors in G. With each point a of G there is associated one such principal subspace whose basis is the set of neighbors of a, distinct from a. This subspace serves to define the local dimension $\dim_a G$ of G at a. Then the global dimension $\dim G$ is $\sup \dim_a G$. The definition of dimension is recursive, completely analogous to the Menger-Urysohn theory, and begins with the null-space which is of dimension -1.

The set function $\dim A$, defined as the dimension of the subspace of G induced by the basis A, is shown to be monotonic and to satisfy the "Summensatz":

$$\dim (A+B) \leqq \dim A + \dim B + 1.$$

It is shown that $\dim G \geqq n$ if and only if G contains $n+1$ distinct points each neighbor to the others. Two subspaces of G are called mutually complementary if every pair of neighbors of G is a pair of neighbors in one and only one of the subspaces. If these are of respective dimensions m and n, then it is proved that $\dim G \leqq 2^{n+m-1} \cdot 3 - 1$.

A variety of continuous (single-valued, neighbor-preserving) transformations $f(G) = G^*$ are defined and relations between them exhibited, for example: (1) inner, if each pair of neighbors in G^* comes from at least one neighbor pair in G; (2) outer, if f^{-1} is one-to-one; (3) folded (svyertivayushchi), if b, b' neighbors in G^* implies that every point of $f^{-1}(b)$ is neighbor to some point of $f^{-1}(b')$; (4) dispersed (razvyertivayushchi), if no two points of $f^{-1}(b)$ are neighbors, for every b in G^*. One interesting theorem is the unique factorization of a continuous transformation into an inner and outer one.

The principal objective of the paper is to establish the invariance of dimension under homeomorphism, and to investigate some types of dimension raising and dimension lowering transformations. *L. Zippin* (Flushing, N. Y.).

Kuratowski, C. Sur un problème topologique de la théorie de la mesure. Colloquium Math. 1, 210–213 (1948).

G. Choquet [same vol., 29 (1948)] raised the following question. In a compact metric 1-dimensional space S, for each finite covering $R=\{G_i\}$ of S by open subsets, let $a(R)=\sum \operatorname{diam}^2(\bar{G}_i-G_i)$. Given $\epsilon>0$, let $a(\epsilon)$ be the greatest lower bound of numbers $a(R)$ for those coverings R all of whose elements have diameter less than ϵ. Does $a(\epsilon)\to 0$ with ϵ? The author shows that if C is a nondense perfect subset of the unit interval I, with positive measure, then the plane continuum $C\times I+I\times C$ provides a negative answer. *G. S. Young* (Ann Arbor, Mich.).

Whyburn, G. T. Continuous decompositions. Amer. J. Math. 71, 218–226 (1949).

Let X be a topological space and G a covering of X by disjoint closed sets. Various types of "continuity" of the collection G are considered, in particular, those which, according to the author, are of interest in complex analysis. Here X would be an open set in the complex plane and conditions involving the compactness of the elements of G are undesirable. The results (difficult to state succinctly) may be translated into propositions concerning interior and related types of transformations but the author feels that the language of decompositions is, in some contexts, both fruitful and natural. *A. D. Wallace* (New Orleans, La.).

Bonhoff, Stéphane, et Colmez, Jean. Sur le problème de Wiener. I. Structures des solutions. Espaces *J*. Revue Sci. 86, 167–169 (1948).

The "problem of (N.) Wiener" is this: given a set E and a group G of one-to-one transformations of E onto itself, to find those topologies T making G the group of homeomorphism of E. This problem is suggested by an article of Wiener [Bull. Soc. Math. France 50, 119–134 (1922); see also Colmez, Revue Sci. (Rev. Rose Illus.) 80, 313–315 (1942); these Rev. 7, 134]. A condition is here formulated, which is necessary if the problem is to have a nondiscrete Hausdorff solution T. Diverse generalizations and particularizations are considered. *R. Arens.*

Colmez, Jean. Sur le problème de Wiener. II. Structures et conditions d'existence de solutions non banales vérifiant certaines conditions. Revue Sci. 86, 170–172 (1948).

Sufficient conditions for the existence of nondiscrete Hausdorff solutions T of Wiener's problem [cf. the preceding review] are formulated. Some are in terms of the existence of suitable filters in E. Two conditions may be quoted here because they involve no concepts introduced ad hoc. (1) There is a completely regular solution if G is transitive but such that, for each x, ${}_xG={}_yG$ for infinitely many y (${}_xG$ is the subgroup leaving x fixed). (2) There are Hausdorff solutions if the number of points fixed under $g\epsilon G$, $g\neq 1$, is finite and uniformly bounded, and ${}_xG$ is countable for some x. *R. Arens* (Los Angeles, Calif.).

Bonhoff, Stéphane. Sur le problème de Wiener. III. Sur les solutions séparées du problème de Wiener. Revue Sci. 86, 173–175 (1948).

[Cf. the two preceding reviews.] The necessary and/or sufficient conditions here considered are of a "local" nature in the sense that they are expressed in terms of the existence of suitable filters of sets at some points. Regular and normal solutions are also considered. *R. Arens.*

Aleksander, Dž. [Alexander, J.]. The connectivity ring of an abstract space. Uspehi Matem. Nauk (N.S.) 2, no. 1(17), 156–165 (1947). (Russian)

Translated from Ann. of Math. (2) 37, 698–708 (1936).

Fox, Ralph H., and Artin, Emil. Some wild cells and spheres in three-dimensional space. Ann. of Math. (2) 49, 979–990 (1948).

A curved polyhedron P in the 3-sphere S is called tame if there is a homeomorphism of S onto itself which transforms P into a Euclidean polyhedron; otherwise P is called wild. The existence of the following objects is established. (1) An arc (=simple arc) whose complement is not simply connected. (2) An arc which is wild even though its complement is an open 3-cell (topologically). (3) An arc whose complement is simply connected but is not an open 3-cell. (4) A wild arc which is the union of two tame ones. (5) A (simple) closed curve which bounds a 2-cell although the fundamental group of its complement is non-Abelian. (6) A wild closed curve which bounds a 2-cell and whose complement has an infinite cyclic fundamental group. (7) A 2-sphere whose exterior is not simply connected. (8) A wild 2-sphere both of whose complementary domains are 3-cells. (9) A 2-sphere whose exterior is simply connected but not a 3-cell. *S. Eilenberg* (New York, N. Y.).

Postnikov, M. M. The structure of the ring of intersections of three-dimensional manifolds. Doklady Akad. Nauk SSSR (N.S.) 61, 795–797 (1948). (Russian)

This paper gives an algebraic characterisation of the homology-ring of a three-dimensional manifold; the coefficient group for homologies is the group of integers mod 2. The author lists as well-known the following properties of the homology-ring $\Delta(M)$ of a three-dimensional manifold M: (1) the additive group Δ is of the form $\Delta^0+\Delta^1+\Delta^2+\Delta^3$, where $\Delta^0=(q)$; $\Delta^1=(u_1)+(u_2)+\cdots+(u_p)$; $\Delta^2=(v_1)+(v_2)+\cdots+(v_p)$; $\Delta^3=e$, and $2q=2u_i=2v_i=2e=0$; (2) the ring is commutative, e serves as identity; (3) if $x\neq e$, then $xq=0$; (4) $uu'=0$ for u, u' in Δ^1; (5) $u_iv_j=\delta_{ij}q$ (Poincaré-Veblen duality); (6) vv' is in Δ^1 if v, v' are in Δ^2. To these the author contributes the following "self-intersection theorem": there exists an element v_0 such that the following identity is satisfied by it and any arbitrary pair of elements of Δ^2: (*) $vvv'+vv'v'=vv'v_0$. The proof is asserted to be straightforward but somewhat tedious. The choice of the element v_0 is explicitly given. In case the manifold M is orientable, the ring being also called orientable, $v_0=0$. However, if M is not orientable, v_0 corresponds to a two-cycle not homologous to zero such that $2v_0$ is a bounding cycle.

The author then sketches the proof that an abstract ring, called an *M.S.*-ring, which satisfies all of the properties listed above, including the self-intersection theorem, is isomorphic to the homology ring of a suitable three-dimensional manifold. This is orientable, or not, depending on whether $v_0=0$. The proof consists in a construction of the manifold M by an inductive process on the rank p of the ring. The induction begins with the three-dimensional sphere in the orientable case and with the three-dimensional "Klein bottle" in the nonorientable. *L. Zippin.*

Wu, Wen-Tsun. Sur le second obstacle d'un champ d'éléments de contact dans une structure fibrée sphérique. C. R. Acad. Sci. Paris **227**, 815–817 (1948).

The problem of choosing a field φ of nonoriented vectors in a bundle of $(n-1)$-spheres over a base space K leads to an obstruction ζ^n, an integral cohomology class of dimension n in K. Let subscripts denote reduction mod 2. It is stated that if W^i are the characteristic classes of the bundle, then $\zeta_2^n=(\xi_2^1)^n+W_2^1(\xi_2^1)^{n-1}+\cdots+W_2^{n-1}\xi_2^1+W_2^n$, where ξ_2^1 is the "coincidence obstacle" of ϕ (defined over K^{n-1}) and another arbitrary field ψ. A proof is sketched. [The result is not clear to the reviewer.] Some remarks are made on fields of 2-elements. *H. Whitney* (Cambridge, Mass.).

Wu, Wen-Tsun. Sur la structure presque complexe d'une variété différentiable réelle de dimension 4. C. R. Acad. Sci. Paris **227**, 1076–1078 (1948).

Let M^4 be an oriented smooth manifold of dimension 4. Imbedding M^4 in a Euclidean space E and transferring back certain well-defined cohomology classes of the Grassmann structure of E gives cohomology classes W^2, W_1^4, W_2^4 in M^4. It is stated (and a proof is sketched) that M^4 admits an "almost complex structure" [C. Ehresmann] if and only if there exists an integral 2-class C^2 which, reduced mod 2, gives W^2, and whose square is $W_1^4+2W_2^4$. In fact, the almost complex structures correspond to such C^2. Hence, for example, the 4-dimensional torus admits an infinity of almost complex structures. *H. Whitney.*

GEOMETRY

***Fano, Gino, e Terracini, Alessandro. Lezioni di Geometria Analitica e Proiettiva.** G. B. Paravia & C., Torino, 1948. vii+642 pp.

The first edition of this substantial treatise appeared in 1930. This second edition contains various small portions of new matter, totalling 12 pages. The most notable addition is a beautifully illustrated account of the curves $y^2=f(x)$, where $f(x)$ is a polynomial of degree 3, 4 or 5.

The book is in five parts. Part I [166 pages] is an introduction to the analytic geometry of the Euclidean plane, including such special curves as epicycloids and the involute of a circle, besides the geometrical interpretation of Fourier series. Part II [129 pages] deals similarly with Euclidean three-space, including vector analysis up to the divergence and curl, as well as a good section on nomography. Part III [164 pages] is on the real projective plane, treated as an extension of the Euclidean plane, with cross ratio used as a link between the synthetic and analytic methods. Collineations (including homologies) and correlations (including polarities) are defined synthetically as by von Staudt, but coordinates are used for the discussion of their invariant and self-conjugate points. Part IV [92 pages] is a thorough treatment of conics, partly synthetic but chiefly analytic. Part V [70 pages] is on real projective space, beginning with collineations and correlations, ordinary polarities and null-polarities, oval and ruled quadrics. The projective theory is followed by metrical considerations, without a clear-cut distinction between the affine and Euclidean geometries. Finally, there is a return to projective geometry for a discussion of line coordinates and linear complexes, with applications to statics.

The book is very readable and moves at a leisurely pace without being long-winded. *H. S. M. Coxeter.*

***Faulkner, T. Ewan. Projective Geometry.** Oliver and Boyd, Edinburgh and London; Interscience Publishers, Inc., New York, 1949. viii+128 pp. 7/6, Great Britain; \$2.35, U.S.A.

This outline of complex projective geometry is an attenuated version of the first two volumes of H. F. Baker's "Principles of Geometry" [Cambridge University Press, 1922]. The subject is approached from two different directions that are not adequately reconciled: the axioms of incidence and Möbius' barycentric calculus. The von Staudt-Hessenberg algebra of collinear points is introduced at a surprisingly early stage. Projectivities, including involutions, are defined in terms of cross ratio, using the "theorem" [page 26] that, if two ranges are in one-to-one correspondence, the cross ratio of any four points of the one range equals the cross ratio of the corresponding four points of the other. This is based on the principle that a one-to-one correspondence between abscissas x and x' implies a bilinear relation, a principle that is contradicted in the real field by the instance $x'=x^3$ and in the complex field by $x'=\bar{x}$. In the chapter on the conic, Pascal's theorem is deduced from Desargues' involution theorem, which is proved by an appeal to that same principle.

In the chapter on absolute elements, a circle is defined as a conic through two arbitrarily chosen absolute points I, J whose join is the absolute line. This definition leads to a concise treatment of coaxal circles, the Euler line of a triangle, axes and foci of conics, the director circle, the auxiliary circle, and the hyperbola of Apollonius. There is also an interesting chapter on the analytic geometry of conics, containing such topics as the harmonic locus and harmonic envelope. The work ends with a brief account of the projective approach to non-Euclidean geometry, with Euclidean geometry as a limiting case.

The general appearance of the book maintains the high standard of the University Mathematical Texts. There is a good index. *H. S. M. Coxeter* (New York, N. Y.).

Kingston, J. M. An icosahedron space. Univ. Washington Publ. Math. **3**, no. 1, 31–34 (1948).

The plane tessellations $\{4,4\}$ and $\{6,3\}$ (of squares, 4 at a vertex, and of hexagons, 3 at a vertex) each admit a translation group that is transitive on the cells, enabling us to associate each point of any cell with a definite point of any other cell. By abstractly identifying such points we derive, in either case, a torus. Analogously, the Euclidean honeycomb $\{4,3,4\}$ (of cubes, 4 at an edge) yields a "hyper-torus" or "cube space." Likewise the spherical honeycomb $\{5,3,3\}$ (consisting of 120 dodecahedra, 3 at an edge, covering the 3-sphere in Euclidean 4-space) admits a group of Clifford translations that is transitive on the cells, namely the binary icosahedral group of order 120 [W. Threlfall and H. Seifert, Math. Ann. **104**, 1–70 (1931), p. 26]. This yields a simply-connected 3-dimensional manifold called the "spherical dodecahedron space" [ibid., p. 66]. The author observes that the same binary icosahedral group is also transitive on the cells of the stellated honeycomb $\{3,5,\frac{5}{2}\}$ which consists of 120 icosahedra, 5 at an edge, covering the 3-sphere four times over [Coxeter, Regular Polytopes, Methuen, London, 1948; Pitman, New York, 1949, pp. 265, 282; these Rev. **10**, 261]. However, as the author admits, the resulting "icosahedron space" is not a

manifold but only a pseudo-manifold: it has a branch point (represented by all twelve vertices of the icosahedron) where the total solid angle is not 4π but 12π.

H. S. M. Coxeter (New York, N. Y.).

***Lauwerier, Hendrik Adolf. Axiomatische Onderzoekingen over de Vlakke Meetkunde.** [**Axiomatic Investigations on Plane Geometry**]. Thesis, Technische Hogeschool te Delft, 1948. 70 pp.

This account of the foundations of geometry achieves a certain simplicity by its restriction to two dimensions and by its use of cross ratio as a primitive concept. In other respects it resembles the classical treatment of Hessenberg [Grundlagen der Geometrie, de Gruyter, Berlin, 1930].

H. S. M. Coxeter (New York, N. Y.).

Bruck, R. H., and Ryser, H. J. The nonexistence of certain finite projective planes. Canadian J. Math. **1**, 88–93 (1949).

Every line of a finite projective plane π contains the same number $n+1$ of points. Tarry [Assoc. Franç. Avancement Sci. C. R. **29**, partie 1, 122–123 (1900); partie 2, 170–203 (1900)] showed that no plane exists for $n=6$. Prior to this paper nothing more could be said about possible values for n, save that for $n=p^r$, p a prime, a Desarguesian plane exists with coordinates from the Galois field $GF(p^r)$; also for various values of the form $n=p^r$, $r>1$, non-Desarguesian planes have been constructed. The principal theorem of this paper proves that if the square-free part of n is divisible by a prime of the form $4k+3$, and if $n\equiv 1$ or 2 (mod 4), then there is no plane with $n+1$ points on a line.

The proof is based on the incidence matrix A of the plane π. Write $B=AA^T=A^TA$. The symmetric matrix B will have $n+1$'s down the main diagonal and 1's elsewhere. Hence if the plane π exists, the matrix B is rationally congruent to the identity matrix. Application of the Minkowski-Hasse theory of rational equivalence of quadratic forms shows that this holds if and only if the Hilbert norm-residue symbol $(-1, n)_p{}^{n(n+1)/2}=+1$ for all primes p. This condition on n is equivalent to that given above.

M. Hall, Jr. (Columbus, Ohio).

Tutte, W. T. The dissection of equilateral triangles into equilateral triangles. Proc. Cambridge Philos. Soc. **44**, 463–482 (1948).

In a previous paper, Brooks, Smith, Stone, and Tutte discussed the decomposition of a rectangle into squares by associating to each such decomposition an electric network [Duke Math. J. **7**, 312–340 (1940); these Rev. **2**, 153]. In this paper, an equilateral triangle Δ is subdivided into equilateral subtriangles. To each such decomposition T of Δ a graph is associated. It consists essentially of the segments connecting the midpoint of each subtriangle with its vertices. The edges of this graph together with the disjoint polygons into which they subdivide the Gaussian plane form the "bicubical map" $M(T)$. As the sides of each of these polygons are parallel to exactly two of the three medians of Δ, the set of these polygons can be divided into three disjoint classes, depending on the nature of their sides. If all the points of each of the polygons of one class are identified, $M(T)$ will degenerate into a 2-complex whose 2-cells are the polygons of the other two classes and all of whose 1-cells are parallel to the same median of Δ. In this way, $M(T)$ determines three 2-complexes. Their edges can be provided with an orientation. The treatment of these "trial 2-complexes" is based on systems of linear equations similar to Kirchhoff's laws for electrical networks. They imply, e.g., that the sides of the subtriangles of Δ are commensurable, and that Δ cannot be decomposed into equilateral triangles no two of which are congruent. It is possible, however, to subdivide Δ into rhombuses and triangles so that no two of these figures have equal sides.

The author also gives an independent definition of a bicubical map and shows that from any such bicubical map one can derive a triangulation of an equilateral triangle. He discusses triangulations of parallelograms (with angles of $\pi/3$ and $2\pi/3$), including as a special case squared rectangles. Finally, he generalizes the main duality theorem of the quoted paper.

P. Scherk (Saskatoon, Sask.).

Andrianov, S. N. A synthetic demonstration of a theorem in Lobachevskian geometry. Učenye Zapiski Kazan. Univ. **101**, kn. 3, 22–23 (1941). (Russian)

Let the four sides of a quadruply-asymptotic crossed quadrangle meet an arbitrary transversal in points A, C, B, D; then BC and DA are congruent segments. The author proves this in the classical manner of Lobachevsky. The following projective proof is simpler. Let the absolute conic meet the transversal in M and N. By Desargues' involution theorem, MN is a pair of the involution $(AB)(CD)$. Hence the two tetrads $MNBC$ and $MNDA$ are projectively related; that is, $BC\equiv DA$.

H. S. M. Coxeter.

Maeda, Kazuhiko. On analogues of Kantor's theorems. Sci. Rep. Tôhoku Imp. Univ., Ser. 1. **32**, 121–131 (1945).

The Miquel-Clifford chain [for a bibliography see M. Venkataraman, J. Indian Math. Soc. (N.S.) **9**, 1–28 (1945); these Rev. **7**, 527] admits of a special case, pointed out by S. Kantor, that if five lines of the chain are tangent to a deltoid, i.e., a tricuspidal hypocycloid, the corresponding Miquel circle reduces to a straight line, the Miquel line; and if six lines of the chain are tangent to the same deltoid, the six corresponding Miquel lines touch another deltoid. T. Kubota "dualized" the Miquel-Clifford chain, replacing the straight lines by cycles, i.e., oriented circles [Monatsh. Math. Phys. **43**, 66–68 (1936)]. In the paper under review the author formulates and proves for the Kubota chain a proposition which is the "dual" of Kantor's theorem. The paper contains no bibliographical references.

N. A. Court (Norman, Okla.).

Wormser, Arthur. Polygons with two equiangular points. Amer. Math. Monthly **55**, 619–629 (1948).

The author shows that a polygon of any number of sides may be constructed which admits of a pair of points analogous to the Brocard points of a triangle. Such a polygon has also the analogues of the Brocard circle and the Brocard ellipse of a triangle. The polygon is necessarily cyclic, the biratio (i.e., the anharmonic ratio) of four of its consecutive vertices is constant, etc. A polygon of a given number of sides is determined by three consecutive vertices. [The author introduces the new terms "nomogon" and its "equiangular points." A custom generally observed in analogous cases would suggest the terms "Brocard polygon" and its "Brocard points." Besides, these terms would serve better both the cause of clarity and the principle of least effort.]

N. A. Court (Norman, Okla.).

Winger, R. M. The parametric treatment of cyclic-harmonic curves. Univ. Washington Publ. Math. **3**, no. 1, 5–14 (1948).

The class of curves represented by $\rho=a\cos(p/q)\theta+k$ (p, q integers; a, k real) were extensively studied by

R. E. Moritz, and by him called cyclic-harmonic curves. The author studies the same curves parametrically, using circular coordinates and introducing the parameter $t=\cos\theta/q+i\sin\theta/q$, which leads to rational expressions for the coordinates. He investigates the Plücker numbers and other geometric properties of the curves, and detects a number of incorrect results in the work of Moritz. He establishes easily that the curves as a class are a special type of trochoids. *R. A. Johnson* (Brooklyn, N. Y.).

Seifert, Ladislav. Über eine Kugelenveloppe. Práce Moravské Přírodovědecké Společnosti [Acta Soc. Sci. Nat. Moravicae] 16, no. 3, 9 pp. (1944). (Czech. German summary)

Die Kugeln, deren Mittelpunkte sich auf einem Kreise $k(0, a)$ befinden und eine Sehne desselben p berühren, umhüllen eine Fläche 8.Grades, welche interessante Krümmungslinien hat und sich als parallele Fläche zu einer von M. Lerch studierten Fläche erweist.

From the author's summary.

Srb, Jan. Polygons of $n+4$ sides inscribed in a rational normal curve of n-dimensional space. Časopis Pěst. Mat. Fys. 73, 93–98 (1948). (Czech)

Bindschedler, C. Zur Elementargeometrie der Ellipse. Elemente der Math. 3, 105–111 (1948).

Gentry, F. C. Three cubic loci. Amer. Math. Monthly 55, 633–635 (1948).

Feld, J. M. Anallagmatic cubics. Amer. Math. Monthly 55, 635–636 (1948).

Goormaghtigh, R. On anallagmatic cubics. Amer. Math. Monthly 55, 636 (1948).

Rossier, Paul. Sur les quartiques gauches. Arch. Sci. Soc. Phys. Hist. Nat. Genève 1, 503–504 (1948).

Elie, Jean. Triangles trihomologiques aux axes d'homologie concourantes. Bull. Math. Phys. Éc. Polytech. Bucarest 10 (1938–39), 49–51 (1940).

Thébault, V. Sur un théorème de M. D. Pompeiu. Bull. Math. Phys. Éc. Polytech. Bucarest 10 (1938–39), 38–42 (1940).

The theorem in question states that the distances from the vertices of an equilateral triangle to a point of its plane are the sides of a triangle.

Thébault, V. Sur l'hexagone inscriptible à côtés opposés parallèles. Bull. Math. Phys. Éc. Polytech. Bucarest 10 (1938–39), 33–38 (1940).

Thébault, V. On the altitudes of the triangle and of the tetrahedron. Amer. Math. Monthly 55, 637–638 (1948).

Thébault, Victor. On the Monge point of the tetrahedron. Amer. Math. Monthly 56, 4–13 (1949).

Blanchard, René, Bouvaist, Robert, et Thébault, Victor. Surfaces cubiques associées à un tétraèdre. C. R. Acad. Sci. Paris 227, 950–952 (1948).

Court, Nathan Altshiller. The tetrahedron and its altitudes. Scripta Math. 14, 85–97 (1948).

Expository lecture.

Reuschel, Arnulf. Konstruktion zweier gleich grosser regulärer Tetraeder, die einander zugleich ein- und umgeschrieben sind. Elemente der Math. 4, 7–11, 25–30 (1949).

Zwikker, C. Anticaustics—a cord construction and a general formula. Philips Research Rep. 3, 466–473 (1948).

The anticaustic of a given caustic is traced by a string construction. One end of the string is wrapped around the shape of the light source while the other is wrapped around the caustic. The scriber, which keeps the string taut, traces the anticaustic. The equivalent of this mechanical construction can be performed graphically or analytically. Examples are given for the analytical construction in the complex plane for a circular caustic and point sources at various locations. *M. Goldberg* (Washington, D. C.).

Krames, Josef. Über allgemeine "gefährliche Raumgebiete" der Luftphotogrammetrie. Monatsh. Math. 52, 265–285 (1948).

In a series of papers on the problem of photogrammetric reconstruction [Monatsh. Math. Phys. 49, 327–354 (1941); 50, 1–13, 65–83, 84–100 (1941); these Rev. 3, 300; 6, 14, 15] the author investigated the cases leading to two or three solutions. In his earlier articles he discussed the surfaces causing this ambiguity and called them "dangerous loci." In the present paper he elaborates once more on his previous work. He combines it with the results of two recent papers [Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. 156, 219–232, 233–246 (1948); these Rev. 10, 205] to define "dangerous regions." *E. Lukacs* (China Lake, Calif.).

Convex Domains, Integral Geometry

Besicovitch, A. S. Measure of asymmetry of convex curves. J. London Math. Soc. 23, 237–240 (1948).

This note proves that if C is a closed convex plane curve enclosing an area A, then inside C there is a closed convex curve with a center of symmetry which encloses an area at least $2A/3$, while around C there is a closed convex curve with a center of symmetry which encloses an area at most $4A/3$. These ratios are attained if and only if C is a triangle.

M. M. Day (Princeton, N. J.).

Minoda, Takashi. On certain ovals. Tôhoku Math. J. 48, 312–320 (1941).

By trigonometric means, it is shown that there exist infinitely many ovals U such that the equiangular n-gons circumscribing U have a constant perimeter. The polar tangential equations of these ovals are given by $p\sim r+\sum(a_k\cos k\theta+b_k\sin k\theta)$ summed for k from one to infinity, excepting $k=mn$, where a_k and b_k and r are constants, θ is the inclination of the tangent to an arbitrary direction and p is the perpendicular distance from the origin to the tangent. The perimeters of different ovals having isoperimetric circumscribing n-gons are also constant. It is shown, also, that when the angle at the moving point P formed by the tangents PA and PB to an oval U is some constant, it cannot be concluded that U is a circle even if the lengths PA and PB are always equal to each other. However, if this condition holds for a continuous infinity of values of the angle P over an interval, then U is a circle.

No references to the work of other authors are given. It is obvious, however, that the author was aware of the work

of Fujiwara on rotors in regular polygons. [See references in a paper by the reviewer in Amer. Math. Monthly **55**, 393–402 (1948); these Rev. **10**, 205.] Also, the last theorem described above was obtained by Meissner [Vierteljschr. Naturforsch. Ges. Zürich **54**, 309–329 (1909)].

M. Goldberg (Washington, D. C.).

Fejes Tóth, László. Inequalities concerning polygons and polyhedra. Duke Math. J. **15**, 817–822 (1948).

The following theorem and its dual, which sharpen previous results of the author [Ann. Scuola Norm. Super. Pisa (2) **13** (1944), 51–58 (1948); these Rev. **9**, 460], are proved. Consider a convex trigonal polyhedron having n vertices with $k_1, \cdots, k_n$ edges meeting at the respective vertices. Let $R_1, \cdots, R_n$ and $r_1, \cdots, r_{2n-4}$ be the distances of an inner point O from the vertices and the faces, respectively. Let further $A(R;k)$ denote the arithmetic mean of the R with weights proportional to the k and $H(r)$ the harmonic mean of the r with equal weights. Then $A(R;k)/H(r) \geqq 3^{\frac{1}{2}} \tan (n\pi/(6n-12))$. Equality holds for the regular trigonal polyhedra with center O. By an example it is shown that $A(R;k)$ cannot be replaced by the arithmetic mean $A(R)$ with equal weights.

W. Fenchel.

Fejes Tóth, L. Über die mittlere Schnittpunktszahl konvexer Kurven und Isoperimetrie. Elemente der Math. **3**, 113–114 (1948).

In the plane let ζ_0 be a fixed convex domain and let ζ traverse all those domains congruent to a fixed convex domain $\bar{\zeta}$ for which $\zeta\zeta_0 \neq 0$. If the number of intersections of the boundaries of ζ and ζ_0 averaged over the above positions of ζ is denoted by $\bar{n}(\zeta, \zeta_0)$, then

$$[LL_0 - 2\pi(F+F_0)][LL_0 + 2\pi(F+F_0)]^{-1} = \tfrac{1}{2}\bar{n}(\zeta, \zeta_0) - 1.$$

H. Busemann (Los Angeles, Calif.).

Rutishauser, Heinz, et Samelson, Hans. Sur le rayon d'une sphère dont la surface contient une courbe fermée. C. R. Acad. Sci. Paris **227**, 755–757 (1948).

Generalizing a theorem of the reviewer [Math. Ann. **101**, 238–252 (1929)] the authors prove the following theorem. Let C be a closed curve on the unit sphere S_n in Euclidean $(n+1)$-space, and let the length of C be $L < 2\pi$. Then on S_n there exists a sphere with spherical radius $L/4$ which contains C. The proof is elementary and very short, and it is valid in a rather general class of spaces (instead of S_n) including Cartan's symmetrical Riemann spaces. [The authors were not quite aware, that the theorem and this proof have already been found by B. Segre [Boll. Un. Mat. Ital. (1) **13**, 279–283 (1934)].]

W. Fenchel (Copenhagen).

Pozdnyak, È. G. Infinitesimal deformation of a cylindrical belt. Uspehi Matem. Nauk (N.S.) **2**, no. 4(20), 170–174 (1947). (Russian)

Let the surface S in E^3 be sufficiently smooth and homeomorphic to a cylindrical belt and such that the boundary curves L_1, L_2 are closed convex curves with nonvanishing curvature in nonparallel planes. If S undergoes a nontrivial infinitesimal (isometric) deformation, then for a given $\epsilon > 0$ at least one L_i contains points with distance less than ϵ whose distance is increased. If $\bar{S}$ originates from S by adding the plane areas bounded by L_1 and L_2 then S is rigid of the first order under infinitesimal deformations of $\bar{S}$.

H. Busemann (Los Angeles, Calif.).

Vidal Abascal, E. Parallel curves on surfaces of constant curvature. Revista Unión Mat. Argentina **13**, 135–138 (1948). (Spanish)

On a surface with constant curvature K let C be a (sufficiently smooth) closed Jordan curve of length L bounding a simply connected domain of area F. Let $x(s)$ represent C with the arc length as parameter. On the geodesic g_s through $x(s)$ which forms with C the fixed angle ω, lay off a fixed distance ρ. The endpoints traverse a curve C_ρ which for $\omega = \pi/2$ is parallel to C in the usual sense. If L_ρ and F_ρ are the length of, and the area bounded by, C_ρ, and $\omega(\rho, s)$ is the angle at which C_ρ intersects g_s, define ω^* by

$$L_\rho \sin \omega^* = \int_0^{L_\rho} \sin \omega(\rho, s) ds_\rho,$$

where s_ρ means arc length on L_ρ. Then

$$(F_\rho - 2\pi K^{-1})^2 + L_\rho^2 \sin^2 \omega^* K^{-1} = (F - 2\pi K^{-1})^2 + L^2 \sin^2 \omega K^{-1},$$

so that the left side does not depend on ρ.

H. Busemann (Los Angeles, Calif.).

Hadwiger, H. Kurzer Beweis der isoperimetrischen Ungleichung für konvexe Bereiche. Elemente der Math. **3**, 111–112 (1948).

A brief proof is sketched for Bonnesen's estimate $L^2 - 4\pi F \geqq (L - 2\pi r)^2$ for convex domains in the plane.

H. Busemann (Los Angeles, Calif.).

Santaló, L. A. On the distribution of planes in space. Revista Unión Mat. Argentina **13**, 120–124 (1948). (Spanish)

If E^3 is divided by planes into regions whose average volume is one, then the average of the squares of these volumes is $4\pi^2/3$. More exactly the procedure is as follows. Let S be the interior of a sphere with radius R. In order that the average of the volumes of the regions into which n planes which intersect S, averaged over all positions of the n planes is one, R must have the value

$$[\tbinom{n}{3}\pi 4^{-3} + \tbinom{n}{2} 3\pi 4^{-3} + (n+1)\tfrac{3}{4}\pi^{-1}]^{\frac{1}{3}}.$$

If $v^2(R)$ is the correspondingly defined average of the squares of these volumes, then $\lim_{R\to\infty} v^2(R) = 4\pi^2/3$.

H. Busemann (Los Angeles, Calif.).

Algebraic Geometry

Severi, Francesco. Il concetto generale di molteplicità delle soluzioni pei sistemi di equazioni algebriche e la teoria dell'eliminazione. Ann. Mat. Pura Appl. (4) **26**, 221–270 (1947).

A courteous and detailed answer to a rather severe criticism by O. Perron of some of Severi's results [Math. Z. **49**, 654–680 (1944); these Rev. **6**, 185]. Most of Perron's criticisms appear to be based upon a misunderstanding of the role played by the principle of the conservation of number in algebraic geometry. Severi also deals with a criticism of Vahlen's example [J. Reine Angew. Math. **108**, 346–347 (1891)] of a rational space quintic curve with a quadrisecant which cannot be obtained as the intersection of three surfaces. Perron has constructed three quintic cones which intersect in this curve [Math. Z. **47**, 318–324 (1941); these Rev. **3**, 302]. On this point honours must be awarded to Perron, since the interpretation of Vahlen's example given by Severi is not one commonly held by algebraic geometers

[cf. Bertini, Introduzione alla Geometria Projettiva degli Iperspazi, 2d ed., Messina, 1923, p. 232, footnote; Baker, Principles of Geometry, v. 5, Cambridge University Press, 1933, p. 211, ex. 5]. *D. Pedoe* (London).

Segre, B. Un nuovo metodo per lo scioglimento delle singolarità. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 411–414 (1947).

Let C be an algebraic curve over an arbitrary field K, lying in S_r, and having no multiple components. Let $S_{r'}$ $(r'=r(r+3)/2)$ and C' be the transforms of S_r and C by the system of quadrics in S_r, and let C_1 be the curve in the Grassmannian of the lines of $S_{r'}$ representing the tangents to C'. The transformation from C to C_1 is birational, and the author states that successive applications of such transformations will eventually yield a curve C_N having no singularities. The main part of the proof consists of an analysis of the effect of the transformations on a branch with parametrization $x=t^n$, $y=ct^{n+m}+\cdots$, $m, n>0$. If K is of characteristic zero every branch has such a parametrization and the proof goes through. But if K has characteristic $p\neq 0$ there can exist branches with no such parametrization; for example, the branch $x=t^p+t^{p+1}$, $y=t^{p^2}$, of the curve $x^{p^2}=y^p+y^{p+1}$. The analysis of the transformation $C'\to C_1$ also breaks down in the case of characteristic p.

R. J. Walker (Ithaca, N. Y.).

Derwidué, L. Essai sur le problème général de la réduction des singularités d'une variété algébrique. I. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 399–412 (1948).

The proof is based on the use of Cremona transformations to lower the index of a singular point of a variety. Let $|\phi_0|$ be the system of first polars of a hypersurface F in S_r. The base points of $|\phi_0|$ are of index zero. Let $|\phi_{k+1}|$, $k\geqq 0$, be the system with no fixed components cut by $|\phi_k|$ on a generic ϕ_k. The base points of $|\phi_{k+1}|$ have index $k+1$. Finally, the generic point of the singular V_{r-2} of F has index -1, and a simple point of F has index -2. Let i be the maximum index of any point of F and let L be the variety whose general point (or points, if L is reducible) is of index i. By a suitable Cremona transformation [see the following review] with L as fundamental variety, F and L are carried into F' and a subvariety A. Let L_1 be the subvariety of A whose generic point has index i on F'. (There are no points on A of index greater than i.) If L_1 is not vacuous the process can be continued, to give $L_2, \cdots, L_t$. Each of the L's is the transform of a variety of dimension $r-i-3$ in a neighborhood of the intersection of two ϕ_{i-1}'s; hence there can be only a finite set of the L's, and so the maximum index is eventually lowered. These definitions and the proof can be extended to general varieties. [The reviewer was unable to follow many of the author's geometric arguments.] *R. J. Walker* (Ithaca, N. Y.).

Derwidué, L. Essai sur le problème général de la réduction des singularités d'une variété algébrique. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 432–444 (1948).

This continuation of the paper reviewed above discusses the Cremona transformations used in reducing the singularities. They are generalized monoidal transformations having L as a fundamental variety. The fundamental elements are obtained, and it is shown that their effect on F, other than that described above, is to leave the singularities unchanged except for the introduction of an ordinary singular variety. The arguments are entirely geometric.

R. J. Walker (Ithaca, N. Y.).

Lo Voi, A. Sulla irregolarità delle superficie multiple cicliche e lo scioglimento della torsione delle superficie algebriche. I, II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 223–228, 228–230 (1947).

The author starts from results given by De Franchis on multiple cyclic varieties [Rend. Circ. Mat. Palermo **48**, 384–388 (1924)] and considers the particular cyclic surfaces which are obtained by adjoining, to the field of rationality of an algebraic surface $f(x, y, z)=0$ having a linear torsion cycle with coefficients $t>1$, the unramified radical (of De Franchis) $u=\{\varphi(x, y, z)\}^{1/t}$, which is multiplied by $e^{2\pi i/t}$ when this cycle is described, but remains invariant when any other cycle is described which, together with the former, forms a fundamental system. He finds that if the original surface f has irregularity q, the cyclic surface has irregularity tq. In the second part the author shows, in the general case of $\sigma\geqq 1$ torsion cycles, that every surface $f(x, y, z)=0$, having torsion, is the image of an involution carried by a surface F (of $(3+\sigma)$-dimensional space) obtained by adjoining, to the field of rationality of the surface f, the radicals (of de Franchis) $u_1, \cdots, u_\sigma$.

M. Piazzolla-Beloch (Ferrara).

Chisini, Oscar, e Manara, Carlo Felice. Sulla caratterizzazione delle curve di diramazione dei piani tripli. II. Ann. Mat. Pura Appl. (4) **26**, 383–388 (1947).

This note extends results previously obtained by the authors [same Ann. (4) **25**, 255–265 (1946); these Rev. **9**, 463]. The authors consider triple planes which have a projective model with an equation (in nonhomogeneous coordinates) of the form $ax^3+3bz^2+3cz+d=0$, where a, b, c, d are general polynomials in x and y whose orders form an arithmetic progression with (nonnegative) common difference $h\neq 1$. [The case $h=1$ is treated in the previous note.] The branch curve of such a triple plane is of even order m possessing a set K of k distinct cusps, and is characterised by the three properties (i) there is an integer $h\geqq 0$ such that $3m^2-16k=12h^2$; (ii) K is contained partially in the complete linear series $|\frac{1}{2}(m-2h)R|$, where R is a set of m collinear points of the curve; (iii) if T is a set of points residual to K with respect to this series, then there is an effective set of points T' equivalent to $T+hR$ and having no points in common with T. Further, if a curve satisfies these conditions the triple planes of which it is the branch curve are birationally equivalent. *J. A. Todd.*

Chisini, O., e Manara, C. F. Sulla caratterizzazione delle curve di diramazione dei piani tripli. Boll. Un. Mat. Ital. (3) **3**, 6–8 (1948).

On donne la ligne général de la démonstration des résultats analysés ci-dessus. *G. Zappa* (Naples).

Manara, Carlo Felice. Per la caratterizzazione delle curve di diramazione dei piani tripli. Boll. Un. Mat. Ital. (3) **3**, 114–119 (1948).

Toute courbe de diramation d'un plan triple ou quadruple peut être complétée d'une composante double de façon à être de la forme $P^3_{2m}+Q^2_{3m}=0$, P, Q étant homogènes de degrés respectifs $2m$ et $3m$. Par application du théorème de Noether $(Af+B\varphi)$ l'auteur montre que ces courbes sont caractérisées par le fait que si R est le groupe de leurs points de rebroussement, et G un groupe de points alignés, on a l'équivalence: $R=mG$. Ce sont également les seules courbes d'ordre $6m$ ayant des rebroussements en tous les points communs à une courbe d'ordre $2m$ et à une courbe d'ordre $3m$. Les démonstrations sont données dans le cas d'une

courbe générique. L'auteur examine ensuite le cas où la courbe présente un autocontact ou un doublet, et se propose d'étudier dans un travail ultérieur le cas où la courbe admet des singularités supplémentaires ou dégénère.

L. Gauthier (Nancy).

Winger, R. M. Binary polars and applications to rational curves. Univ. Washington Publ. Math. **2**, no. 3, 45–53 (1940).

Starting from the proposition that a necessary and sufficient condition that a binary form f should have a k-fold repeated factor is that all the $(k-1)$th polars of f have a common linear factor, the author deduces expressions for the osculating plane and the parameters of the stationary points of a rational space curve defined parametrically. [Reviewer's note. These equations are clearly valid for any curve defined parametrically, assuming appropriate differentiability properties of the functions involved.]

J. A. Todd (Cambridge, England).

Lage Sundet, Knut. Constructions of some rational curves. Norsk Mat. Tidsskr. **28**, 105–108 (1946). (Norwegian)

By $u_1=x_1x_3$, $u_2=-x_1x_3$, $u_3=x_1x_2$ a birational correspondence is defined between the lines u and the points x of a projective plane. An element x or u being given, it is easy to construct the corresponding element. If a curve C of the third class is determined by the double tangent $(0, 1, 0)$ and the ordinary tangents $(1, 0, 0)$ and $(0, 0, 1)$, then its homologue in the above correspondence is a conic. Constructions concerning C (e.g., given a tangent of C, find the point of contact) can now be made with the intervention of this conic. Similar results can be obtained for rational curves of higher classes or degrees.

F. J. Terpstra.

Pompilj, G. Alcuni esempi di superficie algebriche a sistema canonico puro degenere. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 539–544 (1948).

On connaît des exemples de surfaces sur lesquelles le système canonique pur est composé au moyen d'un faisceau de courbes elliptiques [Nöther, Math. Ann. **8**, 495–533 (1875)], ou admet une composante fixe et une partie variable irréductible [Enriques; voir Castelnuovo et Enriques, Math. Ann. **48**, 241–316 (1897)]. L'auteur se propose de montrer par des exemples qu'il existe aussi des surfaces sur lesquelles le système canonique admet une composante fixe et une partie variable elle-même réductible, composée au moyen d'un faisceau de courbes de genre deux. A cet effet l'auteur construit une suite de plans doubles répondant à la question et ayant pour caractères: $P_g=P_a=2h-1$, $p'=p^2+1=8h-9$, $P_2=10h-10$, etc., les systèmes pluricanoniques étant réguliers. La courbe de diramation est une courbe d'ordre $6h$ ayant un point multiple d'ordre $6h-6$ et $4h-4$ doublets. Le système canonique comprend une partie fixe de genre $2h-3$ et de degré -2 et une partie variable formée de $2h-2$ courbes de genre 2 appartenant à un faisceau.

L. Gauthier (Nancy).

Franchetta, A. Sulle involuzioni razionali appartenenti ad una superficie algebrica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 544–549 (1948).

Sur une surface algébrique F, une involution rationnelle I_m d'ordre m donne lieu à une correspondance T dans laquelle deux points sont associés s'ils appartiennent à un même groupe de I_m. Si U est la transformation identique sur F, Severi [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **19**, 751–754, 831–836 (1934)] désigne par (T, U) la série de groupes de points de F qui contient le groupe des points doubles de I_m lorsque ces points sont en nombre fini; lorsque I_m admet une courbe double, la série (T, U) donne l'équivalence fonctionnelle de cette courbe. L'auteur démontre que si l'involution I_m a le genre linéaire relatif p, un groupe G de I_m et un groupe S de la série de Severi sont liés par la relation fonctionnelle $S=(13-p)G-(T, U)$. Cette relation n'était connue [loc. cit.] que dans les cas où I_m est biunivoquement équivalente à un plan ou à une quadrique.

L'auteur démontre ensuite que si l'involution est formée des groupes caractéristiques d'un réseau, la condition nécessaire et suffisante pour qu'une courbe fondamentale pour le réseau le soit aussi pour l'involution I_m est que le degré virtuel v de cette courbe calculé en tenant compte des points bases du réseau soit inférieur à m.

L. Gauthier.

Vaccaro, G. Le superficie razionali prive di curve eccezionali di prima specie. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 549–551 (1948).

Une courbe exceptionnelle E sur une surface algébrique est de première espèce si dans la transformation birationnelle qui la fait correspondre au voisinage d'un point simple, aucun point de E ne se transforme à son tour en courbe exceptionnelle; son genre est alors 0, son ordre -1. De telles courbes n'existent que sur les surfaces réglées et rationnelles. L'auteur détermine les surfaces rationnelles privées de ces courbes exceptionnelles de première espèce: le système représentatif des sections planes $|C|$ possède des points-bases qui doivent appartenir à une courbe fondamentale de genre 0 et d'ordre -1, ne coupant qu'en un point le système résidu et pour laquelle les conditions de passage sont indépendantes. Ces propriétés caractérisent les courbes irréductibles contenues dans l'adjoint de $|C|$. Conforto [Le Superficie Razionali, Zanichelli, Bologna, 1945, p. 209] a ramené ces systèmes à ceux sans courbe fondamentale, à ceux dotés de 2 points bases joints par une droite fondamentale et au système des courbes d'ordre n avec un point $n-r$ et s points r-ples infiniment voisins. Il en résulte que les surfaces cherchées sont les transformées birationnelles sans exception du plan et de la quadrique et les cônes abstraits. Si la surface a le nombre-base 1, elle appartient à ces familles; d'où l'auteur conclut que les seules surfaces rationnelles de nombre-base 1 sont linéaires.

B. d'Orgeval (Grenoble).

Andreotti, A. Questioni di equivalenza relative alle curve riducibili e ai punti base di un fascio di curve sopra una superficie algebrica. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 551–557 (1948).

Andreotti, A. Questioni di equivalenza relative alle curve riducibili e ai punti base di un fascio di curve sopra una superficie algebrica. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 666–671 (1948).

Let H be a pencil (either irrational or rational) of curves on an algebraic surface F, and let H contain k reducible curves C_i, so that

$$C_i=\sum_{s=1}^{h_i} u_s{}^i\Theta_s{}^i, \qquad i=1, 2, \cdots, k,$$

where the $\Theta_s{}^i$'s are the irreducible components of C_i, each one counted $u_s{}^i$ times. The author shows that, if the invariant series S of Severi is defined on F by the pencil H, the

group of points

$$(1)\quad \sum_{i=1}^{k}\sum_{r<s}(u_r^i+u_s^i-1)(\Theta_r^i,\Theta_s^i)+\sum_{i=1}^{k}\sum_{r=1}^{h_i}(u_r^i-1)M(\Theta_r^i)$$

(where $M(\Theta_r^i)$ is a canonical group on Θ_r^i) represents the functional equivalence in S due to the presence in H of the curves C_i. From (1) the author derives some formulas of enumerative geometry, already obtained, in particular cases, by Castelnuovo and Enriques [Ann. Mat. Pura Appl. (3) **6**, 165–225 (1901)] and by Campedelli [same Rend. Cl. Sci. Fis. Mat. Nat. (6) **19**, 781–788, 860–866 (1934)]. Finally, the author states also that the contribution due to a C_i in the calculation of the Zeuthen-Segre invariant, which, according to (1), is given by the number

$$\Delta_i=\sum_{r<s}(u_r^i+u_s^i-1)[\Theta_r^i,\Theta_s^i]+\sum_{r=1}^{h_i}(u_r^i-1)(2\rho_r^i-2)$$

(ρ_r^i being the genus of the Θ_r^i), is always positive, except when either C_i is a multiple elliptic curve (in which case $\Delta_i=0$) or else C_i is a multiple rational curve Θ (in which case Δ_i is less than zero, but the Θ has at least a double point, so that the contribution due to C_i in the calculation of the Zeuthen-Segre invariant is still positive).

F. Conforto (Rome).

Differential Geometry

Efimov, N. V. Qualitative problems of the theory of deformation of surfaces. Uspehi Matem. Nauk (N.S.) **3**, no. 2(24), 47–158 (1948). (Russian)

This article is devoted essentially to the same field as the article of Cohn-Vossen [Uspehi Matem. Nauk **1**, 33–76 (1936)]. The author states that he took into account the Cohn-Vossen paper but that he does not restrict himself to the presentation of results which have appeared since its publication. With the exception of the middle part the style of the article is that of a review of published papers by other mathematicians and the author [the list of papers contains 57 items of which 9 are by Efimov]. Many unsolved problems are carefully indicated.

The first part [54 pages] deals with the problem of realization of an abstractly given metric beginning with the fundamental paper by Weyl and the paper by H. Lewy who completed Weyl's solution. The next sections present A. D. Alexandrov's method based essentially on approximating a curved surface by polyhedral surfaces (method of gluing). The analytical treatment of this method due to Lusternik [whose paper does not seem to have been published] is given in the following section. After this further developments of the gluing method are given and applied to the problem of bending of surfaces, in particular to the bending of a closed convex surface from which a piece has been cut out, and to the problem of realization of a metric given on the infinite plane. The rest of this first part is devoted to the question of uniqueness of realization; one section [written by A. D. Alexandrov] generalizes the theorem of Cohn-Vossen in two directions; the assumption that the surface is closed is replaced by the condition that the total curvature is 4π, and instead of assuming that "the surface is three times differentiable" it is assumed that "the normal curvatures are bounded." The theorem of Alexandrov concerning uniqueness for surfaces of type T (torus) are also strengthened by removing the condition of analyticity. In this part certain as yet unpublished theorems of Pogorelov are given without proof; they assert inflexibility of ovaloids and "caps" assuming boundedness of Gaussian curvature.

The second part [38 pages] deals with infinitesimal deformations and gives a systematic exposition with proofs. It contains earlier results of Liebmann and Rembs and supplements them with additional results. Some of these additions deal with multiply connected surfaces; for such surfaces the existence of a field of rotations does not imply the existence of a field of translations. Efimov introduces the concepts of affine (strong) and projective (weak) rigidity (these two concepts coincide for simply connected surfaces) and interprets them in terms of deformations of the simply connected surface obtained from the given surface by a system of cuts. He also mentions the possibility of intermediate rigidities connected with different subgroups of the Betti group without however being able to prove that corresponding situations actually exist. Truncated ovaloids are also investigated as to their rigidity, and also surfaces of type T and cylindroids, i.e., truncated (noncircular) cylinders; the rigidity character depends in some cases on whether the planes are parallel or not. For the most part the reasoning is based on the assumption of triple differentiability, but the last section of this part presents results of A. D. Alexandrov for surfaces satisfying the less stringent condition of continuous rectifiability.

The exposition in the last part [20 pages] devoted to continuous (finite) bendings "in the small" has again the character of a review of published [and some unpublished] papers. It begins with the Darboux equations and E. E. Levi's result on the connectedness of the set of realizations in the small, assuming that the Gaussian curvature is different from zero, and analyticity. After this, results of Cohn-Vossen, Schilt, Hopf and Efimov are reviewed which deal with pieces of surfaces containing a point where the Gaussian curvature is zero. The local structure of such surfaces is characterized by certain arithmetical invariants such as the "index" of a zero of curvature, and the order of contact with the tangent plane. The discussion centers around the question of stability, or the question of what effect a deformation has on these invariants. The last section deals with relative inflexibility. A family of isometric surfaces is given by a development of z into a series whose terms are homogeneous polynomials of x and y which in general depend on a parameter t. A certain number of terms of the series in the beginning may be independent of t. The index of relative inflexibility is defined in terms of this number. An arithmetical invariant is obtained in a purely algebraic way from the first term of the series and is used to calculate a lower bound (which may be infinity) for the index of inflexibility. *G. Y. Rainich* (Ann Arbor, Mich.).

Efimov, N. On rigidity in the small. Doklady Akad. Nauk SSSR (N.S.) **60**, 761–764 (1948). (Russian)

A surface is said to possess rigidity of the first order if there are no infinitesimal deformations (except trivial ones) which preserve lengths of curves on it up to infinitesimals of the first order. The author points out the difference that exists between the problems of establishing inflexibility and rigidity. In the first case we must prove that a certain solution of a (nonlinear) differential equation cannot be included in a family of solutions; in the second case (which is the one treated in the present note) one proves that a certain (linear) equation has no solutions. It is proved that

the phenomenon of rigidity may exist in the small. More precisely, the author constructs a surface S containing a point P such that every neighborhood of P possesses rigidity of the first order. The surface is analytic and the deformation field is assumed to be analytic. *G. Y. Rainich.*

Pogorelov, A. V. Extension of a general uniqueness theorem of A. D. Aleksandrov to the case of nonanalytic surfaces. Doklady Akad. Nauk SSSR (N.S.) **62**, 297–299 (1948). (Russian)

Let $F(R_1+R_2, R_1R_2, v)$ be a function of R_1, R_2 and the unit vector v with three continuous derivatives and $\partial F/\partial R_1 \cdot \partial F/\partial R_2>0$. Let S be a closed surface in E^3 with two continuous derivatives and positive curvature. If the value $\varphi(v)$ of F at the point $p(v)$ where the exterior normal of S is parallel to v is known (R_1, R_2 are the principal radii of curvature of S at $p(v)$), then S is determined up to translations. *H. Busemann* (Los Angeles, Calif.).

Aleksandrov, A. D. Curves on manifolds of bounded curvature. Doklady Akad. Nauk SSSR (N.S.) **63**, 349–352 (1948). (Russian)

The paper states a number of results, unfortunately without any indications of the proofs. Let M be a two-dimensional manifold with bounded curvature [see A. D. Aleksandrov, same Doklady (N.S.) **60**, 1483–1486 (1948); these Rev. **10**, 147; quoted as A]. A curve on M with initial point p has at p a direction if it forms with itself at p an angle [see A], which then equals 0. Shortest connections issuing from p have directions [see A] at p. All curves considered are assumed to have directions at their end points. The angle of two curves X, Y issuing from p exists and equals the limit of the angle between two shortest connections from p to points $x \epsilon X$ and $y \epsilon Y$ with $x \rightarrow p$, $y \rightarrow p$. In order to take points like vertices of a cone into account, the sector angle between X and Y is defined as the supremum of the sum of the angles formed by X, Y and a finite number of curves issuing from p between X and Y. The sector angles with the same vertex are additive. The total sector angle equals 2π except at an at most countable number of points. The set of directions from p for which no shortest arc with origin p and these directions exists has angular measure 0.

Let L be a simple arc from p to q and Q a geodesic polygon on the "right" of L (except for p and q) bounding with L a domain G. Let α and β be the sector angles of L and Q at p and q measured in G and φ_i the sector angles of Q at its vertices measured in the complement of G. Then the sum $\alpha+\beta+\sum(\pi-\varphi_i)$ approaches a limit as Q approaches L, which is called the right total geodesic curvature (t.g.c.) of L. Both the right and the left t.g.c. of a shortest connection are nonpositive, but need not be zero. If an arc L lies on the boundary of a closed domain G and is a shortest connection of its end points in G, then L has nonpositive t.g.c. toward G. Under an additional hypothesis on the integral curvature of the subregion of G the converse holds for sufficiently small subarcs of the boundary of G. A curve is said to have right geodesic curvature of bounded variation (g.c.b.v.) if for every finite subdivision of L into subarcs the sum of the absolute values of the right t.g.c. stays below a number M, and inf M is the variation of the right t.g.c. of L. If L has right g.c.b.v. then the right t.g.c. is a completely additive function of the arc and L has finite arc length.

If L bounds together with a shortest connection L_0 of its end points a domain G on M which is homeomorphic to a circular disk and τ, τ_0 are the variations of the g.c. of L and L_0 toward G, s and s_0 their arc lengths, and Ω denotes the absolute integral curvature of G [see A] then $s_0 \geqq s \cos(\Omega+\tau+\tau_0)/2$.

Divide L into subarcs L_i. Let $c(L_i)$ be the minimum of the right and left t.g.c. of L_i when both g.c. are positive, the positive t.g.c. of L_i if one of them is nonpositive, and 0 if both t.g.c. of L_i are nonpositive. Then sup $\sum c(L_i)$ for all subdivisions is called the positive part of the t.g.c. of L. Let G be homeomorphic to a circular disk and such that any two of its points can be connected by a shortest arc in M which lies in G, and suppose that the positive part ω^+ of the integral curvature of G [see A] is less than 2π. Then any line s in G (not necessarily simple) of length s satisfies the inequality $s<8p(2\pi-\omega^+)^{-1}[1+\tau^+(2\pi-\omega^+)^{-1}]^2$, where p is the perimeter of G and τ^+ is the positive part of the t.g.c. of L. This result contains as special cases several results of Cohn-Vossen. If a curve L lies in a domain G homeomorphic to a circular disk and $\tau^+ + \omega^+ < 2\pi$ then L either has no double points, or has exactly one double point such that no part of the curve lies within the loop determined by it. Also, estimates for the angle at the double point are given. *H. Busemann* (Los Angeles, Calif.).

Bouligand, Georges. Analyse géométrique et problèmes aux dérivées partielles. Revue Sci. **86**, 223–233 (1948).

Mittleman, Don. The unions of trajectorial series of lineal elements generated by the plane motion of a rigid body. Trans. Amer. Math. Soc. **64**, 498–518 (1948).

The motion of a flat rigid body in a plane can be represented by the motion of a lineal element. The moving lineal element generates a series of lineal elements, and this series is sometimes a union, i.e., a series composed of lineal elements which are all tangent to the base curve of the series. The first part of this paper is devoted to generalities concerning the series of lineal elements generated by the motions of an ordinary flat rigid body (here called a macroscopic body) and two ideal bodies (called, respectively, a microscopic body and an element-particle) in a positional acceleration field. An element-particle is obtained from a macroscopic body by letting the dimensions approach zero in such a way that the centroid, the directions of the principal axes of inertia, and the relative magnitudes of the principal moments of inertia are all kept fixed. The latter part of the paper is devoted to a study of the unions generated by the motion of an element-particle. It is shown that the totality of ∞^5 series of lineal elements generated by the various possible motions can contain at most ∞^2 unions. Various sets of geometrical properties are given which characterize the two-parameter families of unions which can arise in this way. *L. A. MacColl* (New York, N. Y.).

Vivanti, Giulio. Sulle curve piane a normali doppie. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **10(79)**, 256–260 (1946).

Cette note étudie, d'une manière simple et directe, la nature et la représentation analytique des courbes douées d'une infinité continue de normales doubles. L'attention y est plus specialement portée sur le cas où la courbe est constituée par l'ensemble de deux courbes parallèles se rejoignant de façon à former une courbe unique. Cela arrive en particulier lorsque le lieu du milieu du segment déterminé par les points où une même normale coupe orthogonalement la courbe est une courbe fermée admettant un ou un nombre impair de cupsides, et, dans ce cas, l'auteur donne des indi-

cations sur les singularités que peut presenter la courbe à normales doubles envisagée. *P. Vincensini* (Besançon).

Irimescu, Ion. Étude des courbes définies par certaines relations entre leur éléments intrinsèques. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] **3**, 87–94 (1948).

Let there be given a skew curve C and the osculating sphere S at a point O on C. A local coordinate system for C at O is taken as follows: as x-axis the tangent line; as y-axis the normal to C at O tangent to S; as z-axis the line OP, P being the center of S. The latter two lines are called the r-principal normal and r-binormal of C at O. The work is based on a generalization of the Frenet-Serret formulas by O. Mayer. This paper generalizes Bertrand curves in the following manner. A curve is a generalized Bertrand curve if there exists another curve having the same r-principal normal. It is shown that, as in the classical case, there exists a relation between the curvature and torsion, and, in this generalized case, the coefficients involve the angle between the tangents to the curves appearing in the definition. Other generalizations are made, in particular, to curves of Mannheim and of Cesàro. *V. G. Grove.*

Gheorghiu, Gh. Th. Sur les surfaces de Tzitzéica. Bull. Math. Phys. Éc. Polytech. Bucarest **10** (1938–39), 47–48 (1940).

Nous avons déterminé la forme de l'élément linéaire d'une surface Tzitzéica. *Extract from the paper.*

Müller, Hans Robert. Der Drall einer Regelfläche im elliptischen Raum. Monatsh. Math. **52**, 181–188 (1948).

In a previous paper [Monatsh. Math. **52**, 138–161 (1948); these Rev. **10**, 145] the author has investigated the lines of striction of a one-parameter family of curves in 3-dimensional elliptic space. The special results on the ruled surfaces $L=L(u)$ (L denoting a straight line) were derived from the equations expressing the infinitesimal displacement of the canonical tetrahedron attached to $L(u)$. In the paper under review the author completes this study by extending to elliptic space some properties which are classical in Euclidean space: Chasles' correlation and the interpretation of the parameter of distribution $k=k(u)$ on $L=L(u)$ as

$$\lim_{\Delta u\to 0}\frac{\text{distance of } L(u) \text{ and } L(u+\Delta u)}{\text{angle between } L(u) \text{ and } L(u+\Delta u)},$$

Lamarle's formula for the Gaussian curvature at a point X on $L(u)$ by means of $k(u)$ and the distance from X to the central point C of $L(u)$. Due to the existence of two central points on each $L(u)$, two parameters of distribution are defined, the product of which is 1; there are two Hamilton's and Lamarle's equations. [Note by the reviewer: Chasles' correlation actually expresses a projective property and hence is obviously valid in Euclidean elliptic and hyperbolic spaces; only the metric expressions for it are different.] *C. Y. Pauc* (Cape Town).

Kuiper, N. H. On differentiable linesystems of one dual variable. I. Nederl. Akad. Wetensch., Proc. **51**, 1137–1145 = Indagationes Math. **10**, 361–369 (1948).

Let $T=t+\epsilon\bar{t}$ be a (Clifford) dual number ($\epsilon^2=0$) and $\mathfrak{A}(t)$ a dual unit vector defining a ruled surface. The unique dual differentiable continuation of $\mathfrak{A}(t)$ is $\mathfrak{A}(T)=\mathfrak{A}(t)+\epsilon\bar{t}\mathfrak{A}'(t)$. The unit dual vector $\mathfrak{A}(T)$ defines a line congruence (Study's "Synektisches Strahlensystem") called by the author a D-system. If ψ is the dual angle between $\mathfrak{A}(T_0)$ and $\mathfrak{A}(T)$ then for $T\to T_0$ one gets $d\psi=PdT$ ($P^2=(d\mathfrak{A}/dT)^2$), and $S=\int PdT$ is the dual arc length of the D-system. If $\mathfrak{A}(S)$ is a D-system then a ruled surface of it is defined by $S=s(u)+\epsilon\bar{s}(u)$. One gets easily $(dS/du)du=(1+\epsilon\delta)ds$, where $\delta=\bar{s}_u/s_u=d\bar{s}/ds$ is the parameter of distribution of the surface. Hence $\mathfrak{A}(s)$ is a developable surface, and this fact leads easily to a construction of a D-system: such a congruence consists in general of normals to a developable surface which is an envelope of osculating planes of a solid curve (the author points out other exceptional cases too). The last section is devoted to the kinematic interpretation of a D-system. The paper is to be continued. *V. Hlavatý.*

Charrueau, André. Sur les faisceaux de complexes linéaires. C. R. Acad. Sci. Paris **227**, 712–714 (1948).

L'auteur démontre quelques propriétés métriques des ensembles des droites associées aux complexes du faisceau, par exemple du cylindroide de Plücker et du lieu (de la demiquadrique) des droites conjuguées de la droite donnée par rapport aux complexes du faisceau. [Voir K. Zindler, Liniengeometrie mit Anwendungen, t. 1, Leipzig, 1902, pp. 296, 202.] *F. Vyčichlo* (Prague).

Maeda, Jusaku. Some theorems concerning space curves and ruled surfaces. Sci. Rep. Tôhoku Imp. Univ., Ser. I. **30**, 319–362 (1942).

Comme le titre l'indique, ce travail consiste dans l'exposition d'un certain nombre de propriétés relatives aux courbes gauches et aux surfaces réglées de l'espace à trois dimensions. Il débute par une démonstration du théorème suivant lequel les perpendiculaires communes aux cinq couples de droites s'appuyant respectivement sur les différents systèmes de quatre droites extraits d'un ensemble de cinq droites donné coupent orthogonalement une même droite de l'espace. Considérant ensuite une surface réglée quelconque R, l'auteur attache à R d'une part les deux arêtes de rebroussement (adjointes minima de R) des développables engendrées par les deux plans isotropes π et $\bar{\pi}$ issus d'une génératrice quelconque g de R, puis la courbe (dite associée à R) lieu du point (associé à g) milieu du segment determiné par les points des deux adjointes relatifs à une génératrice quelconque g de R. La considération de ces éléments lui permet de caractériser géométriquement certaines surfaces réglées on certaines courbes gauches, en particulier les surfaces réglées à paramètre de distribution constant et les courbes à torsion constante. Il obtient de nouvelles caractérisations de ces dernières courbes en introduisant la plan diamétral d'une courbe gauche (M) en un point M. Ce plan est le lieu des diamètres issus de M des complexes linéaires ayant au moins un contact d'ordre trois avec la développable lieu des tangentes à (M). Il contient, outre le diamètre, la binormale affine et la binormale de Winternitz, cette dernière étant la droite joignant M au point où la parabole cubique osculatrice en M ayant avec (M) un contact du quatrième ordre du moins oscule le plan de l'infini. Les courbes à torsion constante sont celles pour lesquelles le plan diamétral coïncide constamment avec le plan rectifiant de (M), et l'auteur montre que cette condition peut aussi s'exprimer en disant que le diamètre du complexe linéaire osculateur coïncide avec la droite rectifiante de (M). Il effectue ensuite la recherche des courbes pour lesquelles la binormale affine ordinaire et celle de Winternitz sont confondues, et donne les équations intrinsèques de ces courbes sous une forme faisant intervenir les fonctions σ, ζ et la fonction $\wp$ de Weierstrass.

Portant son attention sur le complexe linéaire osculateur L d'une surface réglée R_1 le long d'une génératrice g_1, l'auteur introduit ensuite les notions de centre principal de g_1 (centre de l'involution des couples de points de g_1 dont les plans polaires par rapport à L sont orthogonaux), de plan principal (tangent à R_1 au centre principal), de tangente principale (tangente à R_1 normale à g_1 au centre principal), de courbe et de surface principales (lieux respectifs du centre et de la tangente principales), puis, après une étude du cylindroïde de Plücker faite en relation avec la théorie des nombres duaux, attache à toute génératrice g_1 de R_1 le cylindroïde lieu des axes des complexes linéaires contenant g_1 et trois génératrices consécutives infiniment voisines. Ces notions sont appliquées, d'abord à l'étude de la surface R_1 et des surfaces R_2 et R_3 engendrées respectivement par la normale à R_1 au point central de g_1 et la tangente à R_1 normale à g_1 en ce même point, puis aux développables lieux des tangentes à une courbe gauche quelconque. Les courbes, arêtes de rebroussement des développables pour lesquelles la hauteur des cylindroïdes associés a une valeur constante $|a|$ comprennent les courbes à rayon de torsion T constant ($T=a$), et une deuxième famille, désignée par Γ, caractérisée par la relation $T=a+s^2/a$, s étant l'arc. L'étude de ces dernières courbes conduit à l'énoncé de nombreuses propriétés géométriques.

L'étude de l'ensemble des surfaces R_1, R_2, R_3 donne une propriété caractéristique des surfaces réglées R_1 dont la ligne de striction est asymptotique (R_1, R_2, R_3 doivent avoir même paramètre de distribution sur trois génératrices quelconques correspondentes g_1, g_2, g_3), et fournit un procédé de construction géométrique des surfaces jouissant de la propriété indiquée. En outre, le déplacement du trièdre mobile T attaché à R_1 dont les vecteurs unitaires sont portés par g_1, g_2, g_3 conduit à associer, à chaque génératrice g_1 de R_1, un cylindre de révolution en relation intime avec les trois surfaces R_1, R_2, R_3, dont la définition est la suivante. Soient P un point fixe par rapport au trièdre T relatif à g_1, g une droite quelconque fixe par rapport à T et issue de P. Les ∞^2 droites g engendrent, lorsque g_1 décrit R_1, ∞^2 surfaces réglées, et le cylindre dont il est question est le lieu géométrique des points centraux de ces ∞^2 surfaces réglées relatifs aux ∞^2 génératrices g issues de P. *P. Vincensini.*

Maeda, Kazuhiko. On the osculating Laguerre cycle of the oriented plane curve. Sci. Rep. Tôhoku Imp. Univ., Ser. 1. 31, 55–69 (1942).

Let φ be the angle of an oriented line g with the positive x-axis and p the distance of g from the origin. Put $u=\tan\varphi/2$, $v=\frac{1}{2}p\sec^2\varphi/2$ and $z=u+jv$, $\bar{z}=u-jv$ ($j^2=0$). Then z is the dual coordinate of g. Starting with the equation of a Laguerre cycle $(a_0u+a_1)v=b_0u^3+b_1u^2+b_2u+b_3$ the author investigates first the one-parameter set of Laguerre cycles which have contact of at least the third order with a given oriented curve (M) (along its line $g(0,0)$) and shows that the locus of the singular focus of the Laguerre cycle mentioned above is a circle connected with the affine normal of (M). The envelope of the principal tangent of the Laguerre cycles of the set is in general a parallel curve of an astroid.

Next the author finds the osculating Laguerre cycles to (M) along $g(0,0)$ and gives a necessary and sufficient condition for a superosculating Laguerre cycle in the form of an equation E. If this equation holds good along (M), then (M) itself is a Laguerre cycle. The integral of E leads to the natural equation of a Laguerre cycle:

$$\rho=a(\tan\varphi/2+\tfrac{1}{3}\tan^3\varphi/2)+b$$

(a, b constants, $1/\rho$ the curvature). The last two sections are devoted to similar investigations of curves

$$(\alpha_0u^2+\alpha_1u+\alpha_2)v-(\beta_0u^4+\beta_1u^3+\beta_2u^2+\beta_3u+\beta_4)=0$$

and $v(1-\lambda u)^2=A\lambda^2+B\lambda+C$, where A, B, C are polynomials in u of the fourth order and λ is a parameter.

V. Hlavatý (Bloomington, Ind.).

Kruppa, Erwin. Strahlflächen als Verallgemeinerung der Cesàro-Kurven. Monatsh. Math. 52, 323–336 (1948).

It is known that ruled surfaces have properties similar to those of curves in space. Using this analogy the author discusses ruled surfaces which correspond to the Cesàro curves treated in detail by E. Salkowski [S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1911**, 523–537]. A necessary condition is given which the intrinsic coordinates of a ruled surface must satisfy in order to belong to the class which is analogous to the Cesàro curves. *E. Lukacs.*

Carpenter, A. F. Complexes invariant under reciprocal polar transformations. Univ. Washington Publ. Math. 2, no. 3, 29–32 (1940).

Soit R une surface réglée quelconque de génératrice r, F_1 et F_2 ses deux nappes flecnodales lieux respectifs des tangentes f_1, f_2 aux points flecnodaux situés sur r, Q_1 et Q_2 les quadriques osculatrices à F_1 et F_2 le long de f_1 et f_2, l_1 et l_2 les polaires réciproques d'une droite quelconque l de l'éspace par rapport à Q_1 et Q_2, et l_{12}, l_{21} les polaires réciproques de l_1 et l_2 par rapport à Q_1 et Q_2 respectivement. L'auteur construit, à partir de ces éléments, tout un ensemble de figures géométriques comprenant des complexes linéaires des quadriques et des cubiques dont il étudie des diverses propriétés. Le rôle principal dans cette étude est joué par quatre complexes linéaires se transformant en euxmêmes par polaires réciproques relativement à Q_1 ou Q_2. Trois de ces complexes déterminent une quadrique Q_3 dont le système de génératrices autre que le système commun aux trois complexes appartient au complexe restant. L'intersection de Q_3 et de la quadrique Q osculatrice à R le long de r comprend r et une certaine cubique. L'auteur envisage les correspondances entre points homologues des trois cubiques (dont deux coïncident) transformées de la cubique précédente par rapport aux trois complexes signalés. Il montre que les plans osculateurs à ces trois cubiques en trois points homologues se coupent suivant une droite dont le lieu est la demi-quadrique Q contenant r, puis étudie les dispositions de trois triples de points que les polarités par rapport aux trois complexes ci-dessus déterminent dans les plans osculateurs aux trois cubiques. *P. Vincensini.*

Chang, Su-Cheng. Note on the projective differential theory of plane curves. Tôhoku Math. J. 48, 277–281 (1941).

Of the cubics which have seven point contact with a plane curve C at a point P there exists a pencil which intersects the tangent to C at P in a given point S. This pencil also intersects the osculating conic of C at P in a unique point R other than P. The points R and S are called related points at P. In terms of these points the author derives the following theorem and corollary. The joins of related points at an ordinary point P of a plane curve belong to a pencil whose center lies on the projective normal of C at P. A necessary and sufficient condition that a line l through an ordinary point P of a curve C be the projective normal of C is that there exists a seven point cubic of C at P which not only passes through the points of intersection of the osculating conic C_2 at P of C and the line l but also through the

pole of l with respect to C_2. To geometrically characterize projective curvature and an associated form the author makes use of the points R^* and S^* which are related points associated with that eight point cubic of C at P (other than the cubic of Wilczynski) which has the projective normal for a tangent. Let Q denote the point at which this cubic touches the projective normal. There is only one seven point nodal cubic of C at P whose node is a point N distinct from P on the projective normal. Let P, P_1, P_2 denote the vertices of the canonical triangle of Wilczynski and let M_1, M_2, M_3, M_4 denote the four points in which the tangent to C at the point $P(\sigma+d\sigma)$ intersects the four lines S^*P_1, S^*Q, P_1P_2, P_2R^*. The projective curvature k of C at P is shown to be, except for a constant factor, equal to the cross-ratio (PP_2, NQ). The form $k(d\sigma)^2$, where $d\sigma$ is the projective arc element, is shown to be, except for a constant factor, equal to the cross-ratio (M_1M_2, M_3M_4). *P. O. Bell.*

Marcus, F. Sur une représentation plane des surfaces. Ann. Sci. Univ. Jassy. Sect. I. 30 (1944–1947), 164–170 (1948).

A necessary and sufficient condition that a plane representation of a nonruled analytic surface S exists such that the images of the union-curves of the congruence of projective normals are straight lines is found to be that the congruence of projective normals is a W-congruence. Such a plane representation of the dual union-curves of the reciprocal congruence is found to exist if the reciprocal congruence is a W-congruence. If the union-curves on S of a canonical congruence can be projected from a fixed point into straight lines of a plane, the surface is found to be isothermally-asymptotic. If the congruence is formed by the axes of Čech, the surface is a surface of Čech, for which $\beta_{vv}+3\beta\beta_u=\beta_{uu}+3\beta\beta_v$. If the congruence is that formed by the directrices of Wilczynski, the surface S is a surface of Tzitzeica-Wilczynski, for which $\partial^2 \log \beta/\partial v\partial u=\beta^2+k/\beta$ (k=constant). If the dual-union curves on S of the reciprocal of a canonical congruence can be projected from a fixed point into straight lines of a plane, the surface, in this case also, is found to be isothermally-asymptotic. Surfaces (1) and (2) which correspond in this manner to the congruences of reciprocals of (1) the axes of Čech, and (2) the directrices of Wilczynski are characterized by the equations (1) $\beta_{vv}-3\beta\beta_u=0$, $\beta_{uu}-3\beta\beta_v=0$, and (2) $\partial^2 \log \beta/\partial v\partial u=\beta^2+k_1/\beta$ (k_1=constant), respectively. *P. O. Bell.*

Marcus, F. Sopra una proprietà caratteristica delle congruenze di rette W. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 4, 699–700 (1948).

Suppose one focal surface of a W congruence is referred to its asymptotic curves, and that its defining differential equations have the Fubini form

$$x_{uu}=\theta_u x_u+\beta x_v+px, \quad x_{vv}=\gamma x_u+\theta_v x_v+qx, \quad \theta=\log(\beta\gamma).$$

The second focal surface is the locus of a point $\bar{x}$ whose coordinates are of the form $\bar{x}=\mu x+2(Ax_u+Bx_v)$. Choosing factors of proportionality for $\bar{x}$, A and B, it is shown that the second focal surface has similar defining differential equations for which $\bar{\beta}\bar{\gamma}+\bar{\theta}_{uv}=\beta\gamma+\theta_{uv}$. The purpose of this note is to exhibit this property of W congruences and to show that it is characteristic of W congruences.

V. G. Grove (East Lansing, Mich.).

Rollero, Aldo. Contatto omografico di superficie. Euclides, Madrid 8, 213–216 (1948). (Italian. Spanish summary)

Let F and F' be two surfaces generated by points O, O'. It is said that F is represented on F' by a contact homography of order k if there exists a homography between the spaces of F and F' transforming O into O' and transforming F into F^* having with F' at O' contact of the kth order. Some of the principal properties of this homography are studied. In particular, there exist ∞^3 homographies of the third order between any two surfaces: they are those which transform the coordinate axes and unit point of F' written in the canonical power series expansion of Bompiani into the corresponding coordinate axes and unit point for F. On the other hand if F and F' are projectively applicable such a homography is of the fourth order. *V. G. Grove.*

Salini, U. Sulle trasformazioni puntuali fra due spazi ordinari in una coppia ad Jacobiano nullo di caratteristica uno. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 4, 692–698 (1948).

The author studies the point-to-point (analytic) transformations $T(1)$ between two linear spaces S_3, S_3' in the neighborhood of a couple O, O' of corresponding points at which the Jacobian determinant of the transformation is zero, and of rank 1. By extending the study up to the neighborhood of the second order, it is possible, in the general case, to define a projective coordinate system for each one of the stars with centers at O, O', by means of elements which are intrinsic to the $T(1)$; whence a canonical form for the $T(1)$ is derived; this form puts into evidence that the $T(1)$'s, up to the neighborhood of the second order, depend on seven coefficients, some combinations of which are projective invariants. Several geometric interpretations of these invariants, as well as many other properties of the $T(1)$'s, are indicated. *F. Conforto* (Rome).

Sangermano, Cosimo. Sulle trasformazioni puntuali fra due spazi ordinari. Boll. Un. Mat. Ital. (3) 3, 119–124 (1948).

In this paper the author investigates the point-to-point (analytic) transformations T between two ordinary spaces S_3, S_3', in the neighborhoods of the first and second orders of a couple O, O' of corresponding points at which the Jacobian determinant of T is zero and of rank 1. By means of some lines and planes which are defined intrinsically with respect to T, it is possible to show that there are ∞^3 projective coordinate systems with respect to which the T may be reduced to a typical form. Other lines that are also intrinsic, and some projective invariants of the T, are indicated. A typical form, different from this one, and some other results of this paper were found, simultaneously and independently, by U. Salini [cf. the preceding review].

F. Conforto (Rome).

Longo, Carmelo. Sugli elementi curvilinei piani E_3 con lo stesso E_1. Boll. Un. Mat. Ital. (3) 3, 108–111 (1948).

The aim of this paper is to make clear, with respect to the projective group, the notion of pencil of curvilinear elements E_3 of the third order of a projective plane π, with the same center O and the same tangent t at O: i.e., the totality ∞^1 of the E_3's obtained by linearly combining the equations of two of the given E_3's. The author points out that it is possible to put the given E_3's in a one-to-one correspondence with the points of a projective plane β. A homography Ω of π, leaving O and t invariant, induces a quadratic transformation Q of β into another plane β'. A pencil of E_3's is represented in β by a line r whose Q-transform is a conic. Since for the transformations Q corresponding to the homographies Ω of π a pencil may not be represented by a line, but by a rational curve, it is convenient to call the

pencil a rational pencil. Finally the author shows how it is possible to give a one-to-one correspondence between a pencil of E_3's and a pencil of plane cubics, without making use of the representative plane; also a geometric construction of the generic E_3 of the pencil by means of the two cubics containing the E_3 which define the pencil is indicated.

F. Conforto (Rome).

Sakellariou, Nilos. On a group of contact transformations. Math. Mag. 22, 13–18 (1948).

At the point $P(x^\lambda; \lambda=1, 2, 3)$ of Euclidean three-space, let $t^\lambda(\omega)$ denote a one-parameter family of unit vectors which depend upon the parameter ω. If c is a constant and a_λ^μ are constants which are the elements of an orthogonal transformation then the contact transformations studied by the author are $'x^\lambda = a_\mu{}^\lambda x^\mu + ca_\mu{}^\lambda t^\mu$, $'t^\lambda = a_\mu{}^\lambda t^\mu$. The vectors $t^\lambda(\omega)$ determine a one-parameter family of planes. Let S denote the developable surface of this family and γ its edge of regression. The author determines various invariants of γ under the above contact transformations. For example, it is shown that the ratio of the curvature and torsion of γ is an invariant.

N. Coburn (Ann Arbor, Mich.).

Ortiz Fornaguera, R. On the translation of points in spaces with affine connection. Revista Mat. Hisp.-Amer. (4) 8, 174–191 (1948). (Spanish)

The author rediscovers the properties of Levi-Civita parallelism connected with the corresponding system of differential equations as well as with the "tangential mapping" [see Schouten and Hlavatý, Math. Z. 30, 414–432 (1929); Bortolotti and Hlavatý, Ann. Math. Pura Appl. (4) 15, 1–45, 129–154 (1936) for the general theory].

V. Hlavatý (Bloomington, Ind.).

Sasaki, S. On some properties in the large in the geometry of paths. Tensor 8, 41–53 (1948). (Japanese)

The systems of paths in a two-dimensional manifold defined by a system of differential equations with constant coefficients,

$$\ddot{x} = A_1\dot{x}^2 + 2B_1\dot{x}\dot{y} + C_1\dot{y}^2, \quad \ddot{y} = A_2\dot{x}^2 + 2B_2\dot{x}\dot{y} + C_2\dot{y}^2,$$

are classified into three types under projective transformations of affine connections: $\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j\psi_k + \delta^i_k\psi_j$, and linear transformations of coordinates: $x' = px + qy$, $y' = rx + sy$, $ps - qr \neq 0$. The systems of the first and second types can be brought into the normal form: $\ddot{x} = 0$, $\ddot{y} = \dot{x}^2 \pm \dot{y}^2$. It is first shown that they are both flat projectively but not affinely and that the systems of the third type are all flat affinely. Then the author proves that any two different points can be joined with each other by one and only one path of the system of the second or third type, but not of the first type. He determines also such domains for a point P that any point in the domain can be joined with the point P by one and only one path of a system of the first type.

A. Kawaguchi (Sapporo).

NUMERICAL AND GRAPHICAL METHODS

Uhler, Horace S. Twenty exact factorials between 304! and 401!. Proc. Nat. Acad. Sci. U. S. A. 34, 407–412 (1948).

Exact values are given of $n!$ for $n = 305(5)400$. The other factorials of this fourth century may be found easily by multiplying or dividing the appropriate entry by a six digit number.

D. H. Lehmer (Berkeley, Calif.).

Huckel, Vera. Tables of hypergeometric functions for use in compressible-flow theory. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1716, 13 pp. (1948).

This paper tabulates various hypergeometric functions which arise as particular solutions of Chaplygin's differential equation. The tables should prove useful in the tabulation of other auxiliary functions which may arise in various compressible-flow problems. The adiabatic index for air has been taken as 1.4. [Author's abstract.]

The differential equation concerned is

$$\tau(1-\tau)\frac{d^2Y_k}{d\tau^2} + \{(k+1) - (k+1-\beta)\tau\}\frac{dY_k}{d\tau} + \tfrac{1}{2}\beta k(k+1)Y_k = 0,$$

in which $\beta = 1/(\gamma-1) = 5/2$, and τ is a dimensionless speed variable connected with the Mach number M by the relation $\tau = M^2/(2\beta + M^2)$ or, here, $M = \{5\tau/(1-\tau)\}^{\frac{1}{2}}$. Particular solutions are tabulated for which $Y_k(\tau) \to 1$ as $\tau \to 0$, and give $Y_k(\tau)$ and $-(2/\beta k)dY_k/d\tau$ to 5 and 4 decimals, respectively (the values range from 0 to 10 significant figures), for $k = -15(\frac{1}{2})+15$ and $\tau = 0(0.01)0.5$. The corresponding values of M, from 0.22473 to 2.2361, are also given. Note that the heading $dY_k/d\tau$ in tables 3 and 4 is incorrect; internal evidence indicates that a further factor $-2/\beta k$ is needed.

No provision is made for interpolation, but tables 1 and 3 would appear to be interpolable throughout by means of modified second differences or the "throw-back" [see Interpolation and Allied Tables, reprinted from The Nautica Almanac for 1937, p. 928], both τ-wise and k-wise, except in the corner where $\tau \to 0.5$ and $k \to 0$. Tables 2 and 4 are interpolable τ-wise only with difficulty, owing to a rapid increase in magnitude, and k-wise not at all, as the solutions for $-k$ an integer greater than 1 and $-k$ half an odd integer alternate in character owing to logarithmic singularities at $\tau = 0$ in the former set.

A scarcely adequate introduction accompanies the tables, with reference to works by Garrick and Kaplan [Tech. Rep. Nat. Adv. Comm. Aeronaut., no. 790 (1944); these Rev. 10, 161] and Chaplygin [Učenye Zapiski Imp. Moskov. Univ., Otd. Fiz.-Mat. 21, 1–121 (1904); translated in Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1063 (1944); these Rev. 7, 495], which presumably contain the necessary formulae.

J. C. P. Miller (London).

***Goldstine, Herman H., and von Neumann, John. Planning and Coding of Problems for an Electronic Computing Instrument. Report on the Mathematical and Logical Aspects of an Electronic Computing Instrument, Part II, Volume III.** The Institute for Advanced Study, Princeton, N. J., 1948. iii+23 pp.

[For previous reports cf. these Rev. 9, 208, 622.] Continuing their reports on this subject the authors write, "We wish to develop here methods that will permit us to use the coded sequence of a problem, when that problem occurs as part of a more complicated one, as a single entity, as a whole, and avoid the need for recoding it each time when it occurs as a part in a new context, i.e., in a new problem. The importance of being able to do this is very great. It is likely to have a decisive influence on the ease and the efficiency with which a computing automat of the type that we contemplate will be operable. . . . We envisage that a

properly organized automatic, high speed establishment will include an extensive collection of such subroutines, of lengths ranging from about 15–20 words upwards."

The report contains essentially a detailed description of one possible solution of the problem of having the machine itself process the subroutines from the "library" to produce the final working control tapes. With regard to the form of external storage of the "library" and other data, "we incline towards the use of magnetic wire (soundtrack) as input (and output) medium. We expect to use it at pulse rates of about 25,000 pulses (i.e., binary digits) per second."

R. W. Hamming (Murray Hill, N. J.).

Bloch, R. M., Campbell, R. V. D., and Ellis, M. The logical design of the Raytheon computer. Math. Tables and Other Aids to Computation **3**, 286–295 (1948).

The machine in question is basically a 35 binary machine. Provision has been made, however, for the use of 8 decimal digits externally if desired, with the necessary conversions being done by the arithmetic unit of the machine. The size of the storage blocks, called words, is 45 binary digits, the extra 10 digits being used for algebraic sign, for checks, and blank spaces. A four address code is used for orders. The first two addresses refer to the location of the numbers to be used, the fourth to where the answer is to go, and the third to where the next order is to be found. Each order uses two word spaces and is subjected to the same type of checks as are the numbers themselves.

The main internal memory is to consist of 32 mercury delay lines having a total capacity of 1024 words, though provision has been made for expansion to 4096 words. The pulse repetition rate is 4 megacycles per second. The external memory and means of communicating to and from the machine consists of 4 magnetic tapes which can be scanned at a rate of 500 words per second. Special tape preparing and translating units are separate from the main unit. The arithmetic unit is designed to perform $+$, $-$, $\times$, $\div$. Programs for "double accuracy" and "floating point" arithmetics are discussed in the paper and appear to be rather involved. The speed of the arithmetic unit is not given so that no overall speed of the machine can be figured out.

R. W. Hamming (Murray Hill, N. J.).

Haeff, Andrew V. The memory tube and its application to electronic computation. Math. Tables and Other Aids to Computation **3**, 281–286 (1948).

Greville, Thomas N. E. Recent developments in graduation and interpolation. J. Amer. Statist. Assoc. **43**, 428–441 (1948).

Wayland, Harold. Expansion of determinantal equations into polynomial form. Uspehi Matem. Nauk (N.S.) **2**, no. 4(20), 128–158 (1947).. (Russian)

Translated from Quart. Appl. Math. **2**, 277–306 (1945); these Rev. **6**, 218.

Shimizu, Tatsujiro, and Katayama, Yôichi. Solutions of non-linear equations by punched-card methods. Math. Japonicae **1**, 92–97 (1948).

The authors give a systematic method for solving simultaneous algebraic equations of degrees higher than one, and apply the method to the solution of a nonlinear differential equation of second order. The method used in solving a system of algebraic equations, say $f_i(x, y, z)=0$ $(i=1, 2, 3)$, consists in first finding reasonable bounds on the variables x, y, z. The ranges are then subdivided and the f_i calculated (on IBM equipment) for all combinations of the variables. This provides the values of the f_i at a lattice of points, and various methods of inverse interpolation are available for obtaining a better estimate of the solution x_0, y_0, z_0. If this estimate is not accurate enough the process can be repeated. The method is illustrated by an example of three cubics.

The differential equation used for purposes of illustration is $y''=y^2+4\pi^2 \sin \pi(t-\frac{1}{2})$, $y(-\frac{1}{2})=y(\frac{1}{2})=0$. The solution may be written as

$$y(t)=\int_{-\frac{1}{2}}^{\frac{1}{2}} G(t, s)\{y^2(s)+4\pi^2 \sin \pi(s-\tfrac{1}{2})\}ds,$$

where $G(t, s)$ is the Green's function for $y''=0$, $y(-\frac{1}{2})=y(\frac{1}{2})=0$. Using a five-point Gauss integration formula and noting that $y(t)=y(-t)$ one is led to a system of three cubics which are then solved by a simple modification of the above method.

R. W. Hamming (Murray Hill, N. J.).

Kuntzmann, Jean. Meilleure formule de quadrature approchée à deux valeurs pour les fonctions ayant une dérivée seconde bornée. C. R. Acad. Sci. Paris **227**, 584–586 (1948).

If $|f''(x)|<k$, the approximate formula

$$c^{-1}\int_{-c}^{c} f(x)dx \approx f(-c/2)+f(+c/2)$$

has an error of $0.083kc^3$.

E. Bodewig (The Hague).

Lyusternik, L. Certain cubature formulas for double integrals. Doklady Akad. Nauk SSSR (N.S.) **62**, 449–452 (1948). (Russian)

For a given point $A(x, y)=A[r, \varphi]$ the operator $(D, A)=xD_1+yD_2=rD\cos(\varphi-\theta)$, $\partial/\partial x=D_1=D\cos\theta$, $\partial/\partial y=D_2=D\sin\theta$ gives the relation

$$\iint_Q f(x, y)dxdy=\sum c_i f(A_i)$$

for all polynomials of degree less than n if

$$\iint_Q \exp(xD_1+yD_2)dxdy-\sum c_i \exp(D, A_i)\equiv 0 \pmod{D^n}.$$

The following results are obtained. (1) Taking $A_n{}^i=(\rho, 2\pi i/r)$,

$$S_{\rho, n}=n^{-1}\sum_{i=0}^{n-1} \exp(D, A_n{}^i)\equiv I_0(\rho D) \pmod{D^n},$$

where I denotes the Bessel function. (2) The operator of integration over the unit circle C is $\mathcal{J}_C=2\pi D^{-1}I_1(D)$. (3) The operator of integration over the regular n-gon is calculated. (4) Putting

$$(2\pi)^{-1}\iint_C f(x, y)dxdy=C_0f(0, 0)+\sum_{j=1}^{k} n^{-1}C_j\sum_{i=0}^{n-1} f(\rho_j, 2\pi i/n),$$

then

$$\mathcal{J}_c-C_0-\sum_{j=1}^{k} C_jS_{\rho_j, 4k+2}\equiv 0 \pmod{D^{4k+2}}$$

can be obtained by taking ρ_j as the square roots of the roots of $L_k(x)=0$, where L_k is an orthogonal system of polynomials in the interval $(0, 1)$ with the weight function x, and then calculating the C_j. The results for $k=1, 2, 3$ are given as well as the corresponding formula for integration over the regular hexagon. [The reviewer would like to remark that the numerical results can be obtained in a few lines from the Taylor series.]

E. M. Bruins.

Ditkin, V. A. On certain approximate formulas for the calculation of triple integrals. Doklady Akad. Nauk SSSR (N.S.) 62, 445–447 (1948). (Russian)

The integral of a function f over the volume of the unit sphere is approximated by $\int f dv = \sum M_i f(\rho_i, \theta_i, \varphi_i)$, where ρ, θ, φ are spherical coordinates. A general formula is obtained from

$$J=(4\pi)^{-1}\int f dv=\sum_{k=1}^{p} c_k \sum_{r=1}^{2p} \frac{b_r}{4p} \sum_{r=0}^{p-1} f(\rho_k, \theta_r, \tfrac{1}{2}\pi r/p)$$

correct for all polynomials of degree $4p-1$ if ρ_k are chosen as the positive roots of $S_{2p}(x)=0$, where $S_k(x)$ form an orthogonal set of polynomials with the weight-function x^2 in the interval $(-1, 1)$ and c_k, b_r are the constants obtained from the Gauss approximation:

$$\int_{-1}^{1} \varphi(x)x^2 dx=\sum_{k=1}^{p} c_k[\varphi(\rho_k)+\varphi(-\rho_k)],$$

$$\int_{-1}^{1} \varphi(x)dx=\sum_{r=1}^{p} b_r[\varphi(x_r)+\varphi(-x_r)],$$

x_r the roots of $P_{2p}(x)=0$, $P_k(x)$ being the Legendre polynomials. The author also gives the formulae

$$J=\frac{1}{18} O\left(\sqrt{\frac{3}{5}}\right),$$

$$J=\frac{4}{75} f(0, 0, 0)+\frac{7}{300} I\left(\sqrt{\frac{5}{7}}\right),$$

$$J=\frac{16}{525} f(0, 0, 0)+\frac{27}{1400} I\left(\sqrt{\frac{5}{9}}\right)+\frac{1}{280} D(1),$$

correct, respectively, for polynomials of degree not exceeding 3, 5, 7, where $f(0, 0, 0)$ is the value of the function at the origin; $O(\rho)$, $I(\rho)$, $D(\rho)$ stand for the sums of the values of the function at the vertices of a special octahedron, icosahedron and dodecahedron inscribed in a sphere with radius ρ. *E. M. Bruins* (Amsterdam).

Mikeladze, Š. E. New quadrature formulas and their application to the integration of differential equations. Doklady Akad. Nauk SSSR (N.S.) 61, 613–615 (1948). (Russian)

The author obtains by partial integration of

$$m!R_m=(-1)^m \int_a^b P_m(x)f^{(m)}(x)dx,$$
$$P_m(x)=x^m+c_1x^{m-1}+\cdots,$$

the general formula

$$\int_a^b f(x)dx=\sum_{\nu=1}^{m} \frac{(-1)^{\nu-1}}{m!}[P_m^{(m-\nu)}f^{(\nu-1)}]_a^b+R_m.$$

He gives the polynomials for the quadrature formulas of Simpson, Euler-MacLaurin, Obreschkoff and Petr and calculates the formula using the Čebyšev polynomials. A boundary problem for $y''''=\varphi(x)$ is solved using appropriate polynomials. *E. M. Bruins* (Amsterdam).

Kallmann, Hartmut, und Päsler, Max. Eine neue wellenmechanische Störungstheorie. Ann. Physik (6) 3, 305–316 (1948).

The differential equation under consideration is

$$d^2f/dz^2+S(r)f=0, \quad S(r)=\sum_n a_n r^n.$$

Taking the Laplace transform of the differential equation yields

$$(1) \qquad p^2 g(p)-f'(0)-pf(0)+\sum_n (-)^n a_n d^n g(p)/dp^n=0,$$

$$g(p)=\int_0^\infty e^{-pr} f(r)dr.$$

The unperturbed problem is taken as:

$$p^2 g_0(p)-f'(0)-pf(0)-dg_0(p)/dp=0.$$

The solution for (1) is then obtained by the usual iteration procedure. The method is then applied to the Stark effect in a one-electron atom. *H. Feshbach* (Cambridge, Mass.).

Volpato, Mario. Sulla risoluzione di una particolare equazione integrale lineare di Volterra. Boll. Un. Mat. Ital. (3) 3, 34–40 (1948).

The integral equation

$$\varphi(t)+\int_0^t K(t-\tau)\varphi(\tau)d\tau=f(t)$$

in which $K(z)=\sum_{r=0}^\infty A_r e^{-\alpha_r z}$ with positive A_r, α_r and convergent $\sum_{r=0}^\infty A_r$ can be solved approximately by replacing the kernel by a partial sum of the infinite series [cf., for instance, Whittaker and Robinson, Calculus of Observations, Blackie, London, 1924, § 183]. The author estimates the error and proves the convergence of the method. *A. Erdélyi* (Edinburgh).

Kourganoff, Vladimir. Une solution du problème de Milne par la méthode variationnelle, appliquée à un développement exponentiel de la fonction source. C. R. Acad. Sci. Paris 227, 895–897 (1948).

The variational principle suggested by the author [same C. R. 225, 491–493 (1947); these Rev. 9, 190] for solving the equation of transfer for the case of conservative isotropic scattering is applied with the trial function

$$B(\tau)=\tfrac{1}{2}\int_{-1}^{1} I(\tau, \mu)d\mu=\tfrac{3}{4}F(\tau+Q+\sum a_j e^{-\beta_j\tau}),$$

where F denotes the constant net flux and Q, a_j and β_j are constants to be determined. The solution derived in this manner is compared with other solutions which have been given for the same problem. *S. Chandrasekhar*.

Kourganoff, Vladimir, et Michard, Raymond. Nouvelles solutions variationnelles du problème de Milne. C. R. Acad. Sci. Paris 227, 1020–1022 (1948).

In this paper the variational principle for solving the simplest equation of transfer [see the preceding review] is applied with the trial function

$$(*) \qquad B(\tau)=\tfrac{3}{4}F[\tau+Q+\sum_{j=2}^{\infty} A_j E_j(\tau)],$$

where Q and A_j are constants to be determined and E_j denotes the exponential integral of order j. The constants Q and A_j are then determined by imposing one of two conditions: either by requiring that $\int_0^\infty [F(\tau)-F]^2 d\tau=$ minimum, or that the constant net flux condition is satisfied exactly at certain selected points $\tau=\tau_i$. With the minimal condition the solution obtained with four terms in (*) is $B(\tau)=\tfrac{3}{4}F[\tau+0.710489-0.259739E_2(\tau)+0.323258E_3(\tau)-0.102582E_4(\tau)]$, while with the flux condition at $\tau=0$, 0.05, 0.3, and 1.0 the solution is $B(\tau)=\tfrac{3}{4}F(\tau+0.710582$

$-0.268004E_2+0.369043E_3-0.148342E_4$). Both these solutions appear to be very good approximations.
S. Chandrasekhar (Williams Bay, Wis.).

Bernier, J. Les principales méthodes de résolution numérique des équations intégrales de Fredholm et de Volterra. Ann. Radioélec. 1, 311–318 (1945).
Expository article.

Buš [Bush], V., and Koldvell [Caldwell], S. A new differential analyzer. Uspehi Matem. Nauk (N.S.) 1, no. 5–6(15–16), 113–171 (1946). (Russian)
Translated from J. Franklin Inst. **240**, 255–326 (1945); these Rev. 7, 339.

Born, Maks, Fyurts [Fürth], R., and Prinčl [Pringle], R. V. A photoelectric apparatus for the Fourier transform. Uspehi Matem. Nauk (N.S.) 1, no. 5–6(15–16), 172–174 (1946). (Russian)
Translated from Nature **156**, 756–757 (1945); these Rev. **8**, 56.

Tienstra, J. M. An extension of the technique of the methods of least squares to correlated observations. Bull. Géodésique N. S. **1947**, 301–335 (1947).
The technique of least squares is based on the hypothesis that the observations are not correlated. In reality the observations are always correlated though generally only to a small degree. As the formulae for the case of correlated observations become much more complicated the author is obliged to use the principles of the Ricci calculus, and first finds the law of propagation of errors. Then he proceeds to the discussion of several standard problems of adjustment, for example, that of conditioned observations and that of observations which are explicit functions of parameters. As an application of the second problem he calculates the mean error M_X of the arithmetic mean X of n observations of the same quantity x when the observations are mutually correlated to the same extent. In this case $M_X^2=(c+1/n)m_x^2$, where m_x is the mean error of the quantity x and $c\neq 0$. This formula is in better agreement with the fact that M_X cannot be made arbitrarily small by increasing n.
E. Bodewig (The Hague).

ASTRONOMY

Zagar, Francesco. Sopra due equazioni fondamentali nel calcolo di un'orbita ellittica. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) **2**, 29–46 (1946).
Zagar, Francesco. Qualche modifica al metodo di Gauss per il calcolo di un'orbita ellittica. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) **2**, 95–110 (1946).
Zagar, Francesco. Sul calcolo di un'orbita parabolica. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) **3**, 55–68 (1947).
These three papers constitute a survey of various modifications that may be introduced in the methods of orbit determination by Gauss and Olbers, especially in view of the use of machine methods of calculation that have almost completely replaced the logarithmic method to which many of the older formulae had been adapted. The first paper deals with the fundamental problem of deriving the ratio of sector to triangle, for which new tables are presented. In the second paper the method of determining an elliptic orbit is modified by the introduction of a formula due to Gibbs after a first approximation by the method of Gauss has been obtained. This leads to a convenient computing scheme, presented in detail, that is especially adapted to machine calculation. The third paper gives a revision of the method of Olbers for determining a parabolic orbit. According to the examples furnished, better representation of the observations is obtained with a shorter scheme of computation than that presented in the standard treatises on orbit determination.
D. Brouwer.

Musen, Peter. Über die vektoriell-skalaren Gleichungen der astronomischen Störungstheorie. Z. Naturforschung **2a**, 365–369 (1947).
The elements of planetary motion introduced by Milankowitch [Bull. Acad. Sci. Math. Nat. (A), Belgrade no. 6 (1939)] are represented by the double areal velocity vector normal to the orbital plane, the vector in the direction of the perihelion with absolute magnitude $f(M+m)e$, and the time τ of perihelion passage. The equations for the variations of these elements in perturbed motion were derived by Milankowitch for the components of these vectors. In this paper these equations are derived by retaining the vectorial form throughout. The conditions of osculation are presented in a concise vectorial form, which avoids the use of Lagrangian parentheses. The result of the analysis is that the equations are obtained as algebraic combinations of the conditions of osculation, the integrals of Laplace and the integrals of areas.
D. Brouwer.

Michelacci, Lucia. Sull'integrazione approssimata delle equazioni per il problema dei due corpi di massa variabile. Pont. Acad. Sci. Comment. **8**, 1–12 (1944).
Étant posé le problème astronomique des deux corps de masses variables, on voit qu'il est utile de substituer au rayon vecteur et sa dérivée temporelle l'excentricité ϵ de l'orbite osculatrice et la variable W qui est liée à l'anomalie moyenne u par l'équation de Kepler $2\pi W=u-\epsilon\sin u$. L'auteur applique au système différentiel qui caractérise ϵ et W une méthode d'intégration approchée que D. Graffi a proposé comme une application de la théorie des invariants adiabatiques [Ann. Mat. Pura Appl. (4) **15**, 87–128 (1936)]. La méthode permet aussi de calculer l'époque du passage au périhélie et de donner une limite supérieure de l'erreur commise en substituant aux grandeurs envisagées leurs valeurs approchées. Les formules sont illustrées sur le cas particulier que la somme des masses soit une fonction exponentielle du temps.
G. Lampariello (Messina).

Armellini, G. Sul problema lunare di Hill. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 352–358 (1948).
Il est bien connu que Hill a étudié un cas particulier du problème restreint des trois corps qui trouve une application très importante à la théorie de la Lune. La Terre T étant supposée à l'origine du plan cartésien Txy, on admet que le corps S (Soleil) aît une masse beaucoup plus grande de celle de T et que la droite Tx tourne uniformément autour de T dans un plan fixe coincidant avec Txy. Dans le plan tournant Txy une petite masse L (Lune) se meut sous l'action newtonienne de T et de S. On suppose que la distance $TL=r$ soit petite par rapport à TS. Le problème lunaire de Hill est la recherche des mouvements de L. On a jusqu'ici porté l'attention sur les solutions périodiques du

système différentiel des mouvements de L [cf. Charlier, Die Mechanik des Himmels, t. 2, de Gruyter, Berlin, 1927, § 5]. L'auteur propose dans cette note l'étude des trajectoires de L: par élimination du temps t au moyen de l'intégrale de l'énergie, il parvient à une équation du second ordre entre le rayon vecteur r et l'anomalie de TL. On en déduit une certaine courbe algébrique dont T est un centre de symétrie que l'auteur appelle courbe de séparation parce qu'elle sépare le domaine des éventuels maxima de TL du domaine où l'on peut trouver des minima de TL. Enfin l'auteur montre que la détermination de l'orbite séculaire de L s'obtient par quadratures. *G. Lampariello* (Messina).

Sémirot, Pierre. Chocs triples imaginaires dans le problème des trois corps. Mathematica, Timişoara 23, 85–87 (1948).

This paper is based on an earlier one by Dramba [Mathematica, Timişoara 22, 74–80 (1946); these Rev. 8, 291]. It is shown that Dramba's formulas can be considerably simplified, so that the expansions of the coordinates are given in terms of powers of $t^{\frac{1}{3}}$ and $t^{\frac{1}{3}}\log t$. *W. Kaplan.*

García, Godofredo. The problem of three bodies in the cases of Lagrange and Euler, treated in the general theory of relativity. Summa Brasil. Math. 1 (1946), no. 9, 197–205 (1948). (Spanish)

The author reduces the relativistic equations of motion of three bodies to ordinary equations of classical form, with a modified potential, on the basis of an approximation ascribed to Levi-Civita. It is shown that the triangular solutions of Lagrange and the collinear solutions can be generalized to the case of this potential. *W. Kaplan.*

✱von der Pahlen, E. Einführung in die Dynamik von Sternsystemen. Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften, 10. Astronomisch-Geophysikalische Reihe, Band I. Verlag Birkhäuser, Basel, 1947. 240 pp. 32 Swiss francs; bound, 36 Swiss francs.

Investigations on the mathematical theory of stellar dynamics have centered largely on two problems. The first relates to stellar encounters, their importance for stellar dynamics and the manner in which they are incorporated in the framework of a general theory. The second relates to the consequences for the dynamical theory implied by the known kinematics of stellar motions. The present book is devoted to summarizing and in part extending the results of known investigations on these two problems.

In the treatment of stellar encounters, the author follows an early investigation of Charlier. This method consists in evaluating the energy ΔE exchanged between two stars describing hyperbolic orbits about one another and finding the average rate of increase of ΔE^2. But in the treatments which have followed Charlier's, ΔE is first averaged over some of the parameters of the encounter, squared and then averaged over the remaining parameters. The reviewer cannot see that this procedure gives any meaningful result. It would seem far more reasonable to analyse the effect of encounters on the motion of a star, directly, by finding the mean rate of change of the velocity in the direction of motion and at right angles to the direction of motion. When this is done, it is found that the effect of encounters is to systematically reduce the mean velocity to zero while at the same time making the mean dispersion tend to a finite limit. Treatments of stellar encounters along these lines appeared during the last few years but they were apparently inaccessible to the author.

The treatment of the second major problem in the book is exceptionally complete and satisfactory. As is well known the central problem here is equivalent to one of defining the potential in a dynamical system which admits an integral (other than the energy integral) which is quadratic in the velocities. Detailed investigations of this problem have appeared in the literature. But the author's treatment is more concise and in certain parts more elegant than the published investigations. *S. Chandrasekhar.*

von der Pahlen, E. Über die Entstehung der sphärischen Sternhaufen. Z. Astrophys. 24, 68–120 (1947).

The author attempts to develop a theory for the origin of globular clusters, according to which the formation of a compact spherical cluster is a natural consequence of gravitational collapse in a quiescent star cloud of considerable extent and with little or no initial radial density gradient. A cluster of this sort would have the same density gradient for stars of all masses and would in this respect differ markedly from a cluster in isothermal equilibrium, for which the more massive stars would show a much greater concentration to the center than the less massive ones. The author shows that the times in which the present clusters could have evolved are of the order of 10^8 to 10^9 years and, further, that the presence of small velocities of a more or less random nature, perhaps as large as one to two km/sec, would not materially affect the evolution of the cluster.
B. J. Bok (Cambridge, Mass.).

Reiz, Anders. A perturbation problem in the theory of stellar structure. Ark. Mat. Astr. Fys. 35A, no. 29, 15 pp. (1948).

It is well known that a stellar configuration in radiative equilibrium will be governed by the Lane-Emden function $\theta_3(\xi)$ whenever $\kappa\eta$ is constant (where κ is the coefficient of stellar opacity and $\eta=L(r)M/M(r)L$ is the ratio of the average rate of energy production interior to r to the corresponding average taken over the whole star). Consequently, if $\kappa\eta$ is not strictly constant but varies only slightly from some mean value, it may be expected that the structure of such configuration can be treated by a perturbation method. The outline of such a theory has been given by B. Strömgren and S. Chandrasekhar [Chandrasekhar, An Introduction to the Study of Stellar Structure, Chicago University Press, 1939, chap. 6, § 7]. In this paper, the same problem is considered and the explicit solution is obtained as an integral over two particular solutions of the homogeneous equation $y_{\xi\xi}+3\theta_3^2y=0$. The particular solutions of this equation chosen are those which, respectively, satisfy the boundary conditions (1) $y_1(0)=0$ and $dy_1/d\xi=+1$ $(\xi=0)$ and (2) $y_2(\xi_1)=0$ and $dy_2/d\xi=-1$ $(\xi=\xi_1)$, where ξ_1 is the first zero of $\theta_3(\xi)$. The author further shows that the second solution can be derived from the first according to $y_2(\xi)=cy_1(\xi)\int^\xi\{y_1(t)\}^{-2}dt$, $c=$constant; the constant can be later chosen to satisfy the second of the boundary conditions (2). The solutions y_2 and y_1 are obtained numerically and specific applications of the theory given.
S. Chandrasekhar (Williams Bay, Wis.).

Ževakin, S. A. On the auto-oscillations of a Cepheid model. Doklady Akad. Nauk SSSR (N.S.) 58, 385–388 (1947). (Russian)

The author intends to show that the hypothesis of a thermo-nuclear origin of stellar energy leads to the establishment in the star of radial oscillations which are independent of the initial conditions. Several simplifying assumptions are introduced. The core of the star is supposed to have no mass at all. All the mass is assumed to be in the envelope,

which is extended and isothermal. The equation of motion for such a system is set up using the law of perfect gases and neglecting radiation pressure. The generation of energy through thermo-nuclear processes is supposed to obey Gamow and Teller's equation (with some simplifications), and the loss of energy through radiation is computed using an absorption coefficient of the form $k_1 = k_0\rho^n T^{-s}$. Both loss and gain of energy are equated according to the first law of thermodynamics and a differential equation is obtained which correlates the variation in temperature and radius with all the variables of the problem. This equation must be compatible with the equation of motion previously obtained. By linearizing the system of these two equations and analyzing the characteristic equation with the aid of the Routh-Hurwitz criterion, the conditions for stable equilibrium are found in the form of two inequalities. For commonly accepted star models the first inequality is always satisfied, so that any transitory instability must be of the oscillatory type.

During one period of oscillation the amount of energy gained or lost by a star through thermo-nuclear reactions and radiation is small compared to the total energy of the star. This means that the "perturbing forces" caused by the variation of the total energy are small compared to the forces involved in adiabatic compression and expansion; consequently the method of perturbations can be used in the analysis of the nonlinear system of the two equations of motion. The analysis shows that, for a thermo-nuclear energy following the law $\epsilon = \epsilon_0\rho e^{-\tau}T^{-\frac{1}{3}}$, an infringement of the second criterion of stability causes oscillations with constant amplitude, or "auto-oscillations," as the author calls them. For the case of $\epsilon = \epsilon_0\rho T^m$ the amplitude grows beyond any limit. *L. Jacchia* (Cambridge, Mass.).

Sorokin, V. On the character of instability of isothermal gaseous spheres. Doklady Akad. Nauk SSSR (N.S.) **58**, 1003–1006 (1947). (Russian)

The author derives Eddington's well-known equation governing the first-order radial oscillations for the particular case of an isothermal gas sphere (of infinite mass and radius) and points out that, just as for the gas spheres of finite polytropic index, an isothermal configuration will be incapable of exhibiting small free oscillations if the ratio of specific heats of the constituent gas is less than 4/3, and will be in neutral equilibrium if this ratio is equal to 4/3. *Z. Kopal* (Cambridge, Mass.).

Bondi, C. M. Inward integration of the stellar equations. Monthly Not. Roy. Astr. Soc. **108**, 324–333 (1948).

The present paper deals with the standard problem of the integration of the equations of stellar equilibrium by standard methods. [The author does not, however, seem to be aware that the methods used are, in fact, standard ones and that the problems considered in this paper have been treated more thoroughly and more extensively by L. R. Henrich, Astrophys. J. **93**, 483–501 (1941); **96**, 106–123 (1942); **98**, 192–204 (1943).] *S. Chandrasekhar*.

Kuznecov, E. S. The vertical distribution of the temperature of the atmosphere in radiative equilibrium. Akad. Nauk SSSR. Trudy Inst. Teoret. Geofiz. **1**, 3–94 (1946). (Russian)

The problem which is considered in detail in this paper is essentially that of diffuse reflection and transmission by an isotropically scattering plane-parallel atmosphere with an albedo (for single scattering) less than unity. The method of treatment is based on the integral equation of Schwarzschild-Milne type and follows that given in E. Hopf's "Mathematical Problems of Radiative Equilibrium" [Cambridge University Press, 1934]. The difference, however, is that while Hopf treated the case of a semi-infinite atmosphere, the author is interested in an atmosphere of finite optical thickness. [Exact solutions of some aspects of the problem treated in this paper can be found by different methods, cf. S. Chandrasekhar, Astrophys. J. **107**, 48–72, 188–215 (1948), in particular, pp. 69–72; these Rev. **9**, 593.] *S. Chandrasekhar* (Williams Bay, Wis.).

MECHANICS

Levitskiĭ, N. I. The synthesis of a four-bar linkage for a given trajectory of a point of the connecting rod. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1948**, 1539–1542 (1948). (Russian)

Given a curve, the author proposes to determine approximately a plane four-bar linkage $ABCD$ yielding this curve as the trajectory of a point M of the plane of the connecting rod BC. The problem is restated as follows: A is fixed, B moves on a circumference, M on the given curve, and the parameters are to be determined so that the sum of the squares of the deviations of C from a circle is a minimum. Explicit formulas for the lengths AB, CD and the angle of AD with the x-axis are given in terms of the remaining dimensions, with the aid of summations of certain functions of these variables over the same positions of the mechanism as those used for the least square sums. *A. W. Wundheiler* (Chicago, Ill.).

Facciotti, Guido. Trasformazione di moti nel piano. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **10(79)**, 249–255 (1946).

This note deals with the following problem. Let the rectangular coordinates of a point with respect to a fixed set of axes in a plane be given functions of the time. It is required to find a moving set of rectangular axes such that the coordinates of the point with respect to the moving axes shall also be given functions of the time. It results that the moving axes depend upon an arbitrary choice of sign, and upon one arbitrary function of time. A few of the immediate consequences of the fundamental formulae are developed, and the work is illustrated by two simple examples. *L. A. MacColl* (New York, N. Y.).

Teodoriu, Luca, et Woinaroski, Rudolf. Sur la stabilité de l'équilibre d'un point matériel. Disquisit. Math. Phys. **6**, 137–191 (5 plates) (1948).

A study of the imperfect equivalence of dynamic stability and static stability, or the sign of the work done in a virtual displacement. The systems considered are assumed to be represented by their osculating systems of linear equations with constant coefficients, and lengthy calculations are made for a few typical systems of this type. *P. Franklin* (Cambridge, Mass.).

Rubbert, Friedrich Karl. Vektorielle Ableitung der Zentralbewegung. Astr. Nachr. **276**, 127–129 (1948).

A solution of the two body problem in the notation of complex exponentials. *P. Franklin* (Cambridge, Mass.).

Vâlcovici, Victor. Sur les équations du mouvement d'un solide de masse variable. C. R. Acad. Sci. Paris **228**, 52–53 (1949).

A brief note giving in matric vectorial form equations relating the impulsive forces, moments, velocities and accelerations, for a material object of variable mass. Application is made to the case of a conventional rocket, with its associated reference triedral, reobtaining the equation of G. Lampariello [same C. R. **227**, 35–37 (1948); these Rev. **10**, 72]. *A. A. Bennett* (Providence, R. I.).

Grioli, Giuseppe. Esistenza e determinazione delle precessioni regolari dinamicamente possibili per un solido pesante asimmetrico. Ann. Mat. Pura Appl. (4) **26**, 271–281 (1947).

Grioli, G. Precessioni regolari di un solido pesante asimmetrico. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 420–423 (1948).

Let E denote the central ellipsoid of inertia of a heavy rigid body free to rotate about a fixed point different from the center of gravity. The asymmetrical case, in which no two of the three principal axes of E are equal, is especially considered. Regular nondegenerate precession of such a body is dynamically impossible about a vertical axis, but the author shows that it is possible about any axis making an angle α with the vertical, where α is determined as follows. Let L be the line through the center of gravity and perpendicular to the plane of one of the circular sections of E. Then α is the acute angle between L and the normal to E at a point where L intersects E. The motion is such that L remains perpendicular to the axis of precession, and the period of precession is equal to the period of rotation. *D. C. Lewis* (Baltimore, Md.).

Nadolschi, Victor L. Sur un nouveau cas intégrable de mouvement d'un corps solide autour d'un point fixe. Ann. Sci. Univ. Jassy. Sect. I. **30** (1944–1947), 43–74 (1948).

The three Euler equations for the motion of a rigid body, namely $\dot{p}+(C-B)qr=L$, $\dot{q}+(A-C)rp=M$, $\dot{r}+(B-A)pq=N$, reduce in the symmetrical case, $A=B$, to a single linear differential equation of the form $2f(t)\ddot{p}+f'(t)\dot{p}+2p=g(t)$, where $f(t)$ and $g(t)$ are known functions of t if L, M, and N are known functions of t. This last differential equation reduces to the obviously integrable form $(d^2/d\tau^2+1)p=G(\tau)$ if the new independent variable $\tau=\int[f(t)]^{-\frac{1}{2}}dt$ is used. The paper contains many elaborations and applications of the two statements just made. *D. C. Lewis* (Baltimore, Md.).

Figueras Calsina, E. Distribution of the superaccelerations in a spherical rigid motion. Revista Mat. Hisp.-Amer. (4) **8**, 155–164 (1948). (Spanish)

As it is used in this paper, the term superacceleration (of a point) means the time rate of change of the acceleration of the point. The author considers a rigid spherical shell moving arbitrarily in such a way that the center of the shell remains fixed, and he studies the distribution of the superaccelerations of the points of the shell at an arbitrary instant. Various formulae and theorems are given, relating to the superaccelerations and their components. One of the simplest of the results is the following. If there exist points for which the instantaneous superacceleration vanishes, these are either (1) just two diametrically opposite points, (2) the points of a certain circle, or (3) all of the points of the shell. *L. A. MacColl* (New York, N. Y.).

Artobolevskiĭ, I. I., and Abramov, B. M. Concerning the motion of machines under the action of given forces. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1948**, 1509–1512 (1948). (Russian)

Dealing with systems of one degree of freedom, the authors, with the help of a well-chosen variable, have lengthened the derivation of the energy equation, and discovered one of the more obvious quadrature cases. Careful terminology assures the illusion that the paper pertains specifically to machines. *A. W. Wundheiler.*

Tatevsky, V. M. The characteristic functions and the equations of dynamics. Vestnik Moskov. Univ. **1947**, no. 5, 83–105 (1947). (Russian. English summary)

The author considers the general equations of dynamics in terms of the variable systems $(\dot{p}_i, p_i)$ and $(\dot{q}_i, p_i)$. (The index i will be omitted hereafter.) If H is Hamilton's function, and if $M=H+\sum\dot{p}q$, $K=M-\sum\dot{p}\dot{q}$, then

$$\frac{\partial M}{\partial p}-\frac{d}{dt}\frac{\partial M}{\partial \dot{p}}=0,\quad \frac{\partial K}{\partial \dot{q}}=-\dot{p},\quad \frac{\partial K}{\partial \dot{p}}=q$$

and, if $\partial K/\partial t=0$, then $\partial\dot{K}/\partial\dot{q}=-\ddot{p}$, $\partial\dot{K}/\partial\dot{p}=\dot{q}$. The variational principles $\delta\int Mdt=0$, $\Delta\int\dot{p}\partial M/\partial\dot{p}dt=0$ hold. The variation Δ is isoenergetic.

The analogues of the various forms of dynamic equations (except the Hamilton-Jacobi one) follow, all expressed in the new variables, with no particularly simple forms appearing. The theory of small oscillations in the variables p, $\dot{p}$ is developed in complete parallel to its familiar form. A case of order reduction, $M=p^2/2m+F(\dot{p})$, for one degree of freedom, imitates the harmonic oscillator $L=\dot{q}^2/2m+F(q)$. The exposition is elementary and detailed. *A. W. Wundheiler* (Chicago, Ill.).

de Donder, Théophile, et Melchior, Paul. Le principe de moindre contrainte de Gauss appliqué à la dynamique des corps solides à liaisons non holonomes. C. R. Acad. Sci. Paris **227**, 1017–1018 (1948).

The equations of motion of a conservative nonholonomic system are obtained from the principle of least constraint (due to Gauss) with the apparent purpose of showing a connection with a modified least action principle. *D. C. Lewis* (Baltimore, Md.).

Hydrodynamics, Aerodynamics

Manarini, M. Sui paradossi di D'Alembert e di Brillouin nella dinamica dei fluidi. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 427–433 (1948).

Consider a rigid body moving uniformly on a helix through inviscid compressible fluid, which extends to infinity in all directions. It is proved that d'Alembert's paradox, absence of direct resistance, subsists even when the fluid contains vortex sheets which do not extend to infinity. The proof is simple and is based upon a vectorial application of Gauss's theorem, together with the hypothesis that the mean values of ρvr^3 and $(p-p_0)r^2$, over a sphere of radius r, tend to zero as $r\to\infty$. In like manner it is proved that Brillouin's paradox holds, that the fluid must contain regions of negative pressure. The author concludes that neither paradox can be avoided in the absence of an infinite wake. *L. M. Milne-Thomson* (Greenwich).

Manarini, Mario. Sulle equazioni della dinamica dei fluidi perfetti. Boll. Un. Mat. Ital. (3) 3, 111–114 (1948).

An identity relating the acceleration vector in a continuous medium to the momentum transfer tensor $\rho v^i v^j$ was noticed by Greenhill [article Hydromechanics, Encyclopaedia Britannica, 9th ed., 1875]. Employing this identity and introducing a four dimensional treatment, von Laue [see, e.g., Tolman, Relativity, Thermodynamics, and Cosmology, Oxford, 1934, §§ 36–38] constructed an energy-momentum tensor T^{ij} such that Cauchy's equations of motion and the d'Alembert-Euler continuity equation, suitably corrected for relativistic effects, assume the form $T^{ij}{}_{,j}+\rho f^i=0$, whence momentum theorems follow immediately. These results remain valid a fortiori in the special case when relativistic effects are neglected and when the stress reduces to a pure pressure. Pailloux [C. R. Acad. Sci. Paris 225, 1122–1124 (1947); these Rev. 9, 389], without mentioning prior work on the subject, derived ab ovo the equations appropriate to this special case. The present author derives Pailloux's results in dyadic notation.

C. Truesdell (Washington, D. C.).

Stoker, J. J. Waves over beaches of small slope, under a dock, under an overhanging cliff, and past plane barriers. Communications on Appl. Math. 1, 101–108 (1948).

Friedrichs, K. O. Water waves on a shallow sloping beach. Communications on Appl. Math. 1, 109–134 (1948).

Isaacson, Eugene. Waves against an overhanging cliff. Communications on Appl. Math. 1, 201–209 (1948).

Friedrichs, K. O., and Lewy, Hans. The dock problem. Communications on Appl. Math. 1, 135–148 (1948).

John, Fritz. Waves in the presence of an inclined barrier. Communications on Appl. Math. 1, 149–200 (1948).

The paper by Stoker gives a summary of the linear wave theory and of the other papers. The problem of two dimensional progressing waves over beaches with slope angle $\pi/2n$, n an integer, has been discussed recently by Bondi, by Miche [Ann. Ponts Chaussées 1944 (114e année), 25–78, 131–164, 270–292, 369–406; these Rev. 7, 348], by Lewy [Bull. Amer. Math. Soc. 52, 737–755 (1946); these Rev. 9, 163] and by Stoker [Quart. Appl. Math. 5, 1–54 (1947); these Rev. 9, 163]. All of the solutions given by these authors become more complicated as n becomes larger. Friedrichs finds an approximate representation of the solution for large values of n. To derive it the solution is first obtained for integer n in the form of a single complex integral, which can be treated by the saddle point method for large values of n. The paper by Isaacson discusses, by the method of Lewy, a special case of waves in the presence of an overhanging cliff. It is shown that the amplitudes behave in some respects in the opposite way as compared with the case of sloping beaches. The paper of Friedrichs and Lewy gives an explicit solution for the dock problem over a fluid of infinite depth. The solution is given by the sum of two integrals of Laplace type taken over a complex path of integration. The paper by John deals with the effect of a plane rigid barrier on the surface waves. The barrier is inclined at an angle $\pi/2n$ and extends either from the surface down into the water for a finite distance or extends from infinity up to a certain distance below the surface. In the second case a uniquely determined reflection coefficient is determined by stipulating that there should be no progressing wave coming towards the barrier from the opposite side.

A. Weinstein (College Park, Md.).

Ertel, Hans. Ein neues Verfahren zur Konstruktion von Trajektorien in Strömungsfeldern. Z. Angew. Math. Mech. 28, 270–274 (1948). (German. Russian summary)

The paper describes a step-by-step integration procedure for finding the trajectory of a fluid particle in an unsteady field of flow. Both numerical and graphical methods are discussed.

C. C. Lin (Cambridge, Mass.).

Beach, James W. Flow of viscous fluid between slowly rotating eccentric cylinders. Iowa State Coll. J. Sci. 23, 7–10 (1948).

This paper deals with the problem of the slow, steady motion of an incompressible viscous fluid between two eccentric, rotating, infinitely long cylinders. As the stream function satisfies a biharmonic equation, the general solution can be expressed in terms of analytic functions of a complex variable. The author then maps the region between the eccentric circles on the region between concentric circles and solves the boundary-value problem in the transformed plane. The force and torque on either cylinder are given and discussed. Of course, by different methods this problem has already been treated by many authors, for instance G. B. Jeffery [see H. L. Dryden, F. D. Murnaghan and H. Bateman, Bull. Nat. Res. Council no. 84 (1932), pp. 232–242].

Y. H. Kuo (Ithaca, N. Y.).

Howell, A. R. A theory of arbitrary aerofoils in cascade. Philos. Mag. (7) 39, 913–927 (1948).

Using four successive conformal mappings, the irrotational flow of an incompressible fluid through a cascade of obstacles has been "transformed" into the flow about a circular obstacle. The first three mappings which transform the cascade into a "near circle" have been used for this purpose by several authors and the final mapping function is an exponential with a Laurent series for its argument. The author discusses briefly a particular example.

G. F. Carrier (Providence, R. I.).

Meksyn, D. Integration of the boundary-layer equations for a plane in a compressible fluid. Proc. Roy. Soc. London. Ser. A. 195, 180–188 (1948).

The author applies his asymptotic method of integration for boundary layer equations in an incompressible fluid to the case of a compressible fluid. The results obtained by some simple calculations are in close agreement with those obtained by the numerical calculations of Emmons and Brainerd for certain specific values of Prandtl number and Mach number.

C. C. Lin (Cambridge, Mass.).

Meksyn, D. The laminar boundary-layer equations of bodies of revolution. Motion of a sphere. Proc. Roy. Soc. London. Ser. A. 194, 218–228 (1948).

The author derives again the boundary layer equations for axially symmetrical flows past bodies of revolution, and applies them to the front part of the sphere. In this process, the author drops certain terms from his equation without giving a detailed justification. The results, however, appear to agree with Fage's computations and measurements.

C. C. Lin (Cambridge, Mass.).

Görtler, H. Ein Differenzenverfahren zur Berechnung laminarer Grenzschichten. Ing.-Arch. 16, 173–187 (1948).

The equations for the flow in a laminar boundary layer are replaced by standard difference equations. A convenient method for arranging the calculations is developed and the boundary layer profiles for several particular cases are

evaluated numerically. For the semi-infinite flat plate without pressure gradient the numerically computed results agree within 3% of the exact values given by Howarth [Proc. Roy. Soc. London. Ser. A. **164**, 547–579 (1938)]. For the similar problem but with a pressure gradient imposed by a linearly decreasing free stream velocity, the results are in agreement with those of Howarth and considerably better than those of Pohlhausen. Calculations are given for the boundary layer on a circular cylinder. The extension to the boundary layer on a body of revolution and to problems of boundary layer suction and discharge is also discussed. *F. E. Marble* (Pasadena, Calif.).

Wieghardt, K. Über einen Energiesatz zur Berechnung laminarer Grenzschichten. Ing.-Arch. **16**, 231–242 (1948).

The author employs an integral equation for the energy flux in an incompressible boundary layer as well as the integral form of the momentum equation and defines an energy defect thickness of the boundary layer in analogy to the conventional displacement and momentum thicknesses. This additional equation allows the use of a two-parameter family of boundary layer profiles in approximate solutions instead of the conventional one-parameter family. The parameters chosen are $\lambda^* = (\delta_2^2/\nu)dU/dx$, $\epsilon = (\delta_2/U)\partial u/\partial y|_0$, where δ_2 is the momentum thickness of the boundary layer, U the free stream velocity, u the boundary layer velocity parallel to the surface, ν the kinematic viscosity, and x, y the coordinates parallel and normal to the surface, respectively. The first parameter is conventional while the second represents a dimensionless form of the slope of the boundary profile at the surface and is consequently related to the shear distribution. The author discusses a numerical method for computing the variation of these parameters in the direction of flow when an initial boundary layer profile and the free stream velocity distribution are given, which amounts essentially to a stepwise integration of the simultaneous differential equations resulting from the energy and momentum integral expressions.

An approximation using a polynomial

$$u/U = 1-(1-y/\gamma)^2(1+A_1y/\gamma+A_2(y/\gamma)^2+A_3(y/\gamma)^3)$$

is considered [see Mangler, Z. Angew. Math. Mech. **24**, 251–256 (1940); these Rev. **10**, 75] and a two parameter analysis is carried out for the cases of a linearly decreasing free stream velocity, a circular cylinder, and an elliptic cylinder. The results compare favorably with more nearly exact solutions and experimental results.

F. E. Marble (Pasadena, Calif.).

Walz, A. Anwendung des Energiesatzes von Wieghardt auf einparametrige Geschwindigkeitsprofile in laminaren Grenzschichten. Ing.-Arch. **16**, 243–248 (1948).

The integral form of the energy equation for the laminar boundary layer used by Wieghardt [see the preceding review] is employed in the calculation of a single parameter family of velocity profiles. The introduction of this energy relation in addition to the integral form of the momentum equation permits the author to delete the boundary condition which follows from the boundary layer equation:

$$\frac{dU}{ds}\frac{\delta_2^2}{\nu} = -\left.\frac{\partial^2(u/U)}{\partial(y/\delta_2)^2}\right|_{y=0},$$

where U and u are the free stream and local velocity components parallel to the surface, δ_2 the momentum thickness of the boundary layer, ν the kinematic viscosity and s and y the coordinates parallel and normal respectively to the surface. A set of two first order differential equations results and a convenient arrangement of their numerical solution is devised. The author applies his method to the boundary layer in an adverse pressure gradient resulting from a linear decrease of free stream velocity parallel to the surface and to the boundary layer of an elliptic cylinder. The results are in fair agreement with those of Howarth and of Wieghardt. *F. E. Marble* (Pasadena, Calif.).

Pretsch, J. Grenzen der Grenzschichtbeeinflussung. Z. Angew. Math. Mech. **24**, 264–267 (1944).

Boundary layer profiles are considered under conditions of continuous boundary layer removal and of continuous boundary layer exhaust; solutions are obtained for the limiting cases of very strong removal and very strong exhaust. For strong removal the profile is found to be independent of the imposed pressure gradient and is of the form $u/U = 1-e^{v_0 y/\nu}$, where $-v_0$ is the large suction velocity at the surface. For continuous high velocity boundary layer exhaust it is demonstrated that the asymptotic solution is independent of the viscosity. This result allows similarity solutions only for particular distributions of exhaust velocity at the surface. A variety of these asymptotic profiles are computed and the results given in graphical form.

F. E. Marble (Pasadena, Calif.).

Riegels, F., und Zaat, J. A. Zum Übergang von Grenzschichten in die ungestörte Strömung. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. **1947**, 42–45 (1947).

The authors consider the nonlinear ordinary differential equation describing the similarity solution for the boundary layer equations when the free stream velocity varies as a power m of the distance measured from the leading edge along a semi-infinite flat plate: $f'''+ff''+2m(m+1)^{-1}(1-f'^2)=0$. The differential equation is linearized in the region where the boundary layer velocity approaches that of the free stream by noting that in this region the stream function varies almost linearly with the distance from the plate. The equation becomes $f'''+(\eta-\Delta)f''-2m(m+1)^{-1}(f'-1)=0$, where Δ is a constant, and has solutions for f' of the confluent hypergeometric type. These solutions correspond to the "outer solutions" of von Kármán and Millikan. When the exponent of the free stream flow has the value of a half-integer the hypergeometric series becomes a polynomial and the solutions are easily calculable. The solutions so obtained are reasonable even for small distances from the plate and consequently the arbitrary constant remaining in the solution is obtained through equating the corresponding value of the displacement thickness with the known value of Hartree. The velocity distributions obtained in the region approaching the free stream condition are in good agreement with Hartree's results and the authors derive a Lagrangian interpolation formula which they use to obtain solutions for values of m other than those explicitly computed.

F. E. Marble (Pasadena, Calif.).

Viguier, G. Quelques remarques sur la couche limite de Prandtl. Son équation dans le cas de gradients de vitesse élevés. Recherche Aeronautique **1948**, no. 1, 7–9 (1948).

The author considers the boundary theory with his nonlinear relation [C. R. Acad. Sci. Paris **224**, 713–714 (1947)] between the viscous stress and the rate of strain for the case of extremely large velocity gradients. *C. C. Lin.*

Tetervin, Neal. Remarks concerning the behavior of the laminar boundary layer in compressible flows. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1805**, 20 pp. (1949).

Comolet, Raymond. Calcul de l'épaisseur de la couche limite dans une tuyère convergente de révolution. C. R. Acad. Sci. Paris **226**, 2049–2051 (1948).

The von Kármán integral equation adapted for the body of revolution by C. B. Millikan is employed to calculate the boundary layer within a convergent axially symmetric channel. It is assumed that the boundary layer velocity profiles are everywhere similar and the shear coefficient varies as a power of the Reynolds number based upon the boundary layer thickness. Using essentially the fact that the displacement thickness exceeds the momentum thickness, the author infers the validity of his solution for sufficiently large radii. *F. E. Marble* (Pasadena, Calif.).

Wang, Chi-Teh. Variational method in the theory of compressible fluid. J. Aeronaut. Sci. **15**, 675–685 (1948).

Bateman [Proc. Roy. Soc. London. Ser. A. **125**, 598–618 (1929)] showed that the equations of steady irrotational motion of an inviscid compressible fluid are obtained by making the volume integral of the pressure a maximum. The present author undertakes to apply the Rayleigh-Ritz method to obtain approximate solutions to boundary-value problems. There are certain difficulties in applying Bateman's theorem to infinite domains; however, these are solved by modifications of the integral involved. The first example treated is a circular cylinder. For other cylinders, conformal mapping can be used, even though the potential is not harmonic. The case of the circular cylinder is carried out in detail. For convenience it is necessary to replace $\gamma/(\gamma-1)$ by an integer, where γ is the adiabatic exponent; therefore numerical work is carried out for $\gamma=2$. The results are compared with those obtained by other procedures. The author states that a similar calculation by Braun [Ann. Physik (5) **15**, 645–676 (1932)], approximating to the linearized equations of motion, is in error. *W. R. Sears.*

Neményi, P., and Prim, R. Erratum: "Some properties of rotational flow of a perfect gas." Proc. Nat. Acad. Sci. U. S. A. **35**, 116 (1949).

The paper appeared in the same Proc. **34**, 119–124 (1948); these Rev. **9**, 476.

Hicks, B. L. On the characterization of fields of diabatic flow. Quart. Appl. Math. **6**, 405–416 (1949).

This is the continuation of a previous paper on the "diabatic" flow of a compressible fluid [same vol., 221–237 (1948); these Rev. **10**, 160]. The author now proposes to study other types of irrotational diabatic flow characterized by a vector function $\mathbf{N}=\mathbf{V}/(gRT)^{\frac{1}{2}}$. Here $\mathbf{V}$, T, and R are respectively the velocity, temperature and the gas constant; and g is chosen to be a scalar function of N. It is found that by proper choice of $g(N)$, the entire flow field can be made either elliptic, parabolic or hyperbolic. To illustrate the wide variety of such flows, four general types are discussed. *Y. H. Kuo* (Ithaca, N. Y.).

Meyer, R. E. The method of characteristics for problems of compressible flow involving two independent variables. I. The general theory with a note on the calculation of axially-symmetrical supersonic flows by S. Goldstein. Quart. J. Mech. Appl. Math. **1**, 196–219 (1948).

This is a review of the subject indicated by the title, which is attributed to Massau [Annales de l'Association des Ingénieurs Sortis des Écoles Spéciales de Gand (1) **12**, 185–444 (1889); (2) **23**, 95–214 (1900)], Busemann, Prandtl, Temple, Sauer, and Guderley. The author presents his own derivation of the theory, which is similar to the more general discussion of Courant and Hilbert [Methoden der Mathematischen Physik, v. 2, Springer, Berlin, 1937]. It is applicable to both plane and axi-symmetric steady flow and to one-dimensional, axi-symmetric, or spherically-symmetric unsteady flow. In each case the result is a step-by-step numerical method of calculation. The paper ends with two appendices. In the first, the question of the regularity of the orthogonal coordinates introduced is considered. The second, contributed by S. Goldstein, is a note on the calculation of axi-symmetric supersonic flows when the velocity is given along the axis. *W. R. Sears* (Ithaca, N. Y.).

Lighthill, M. J. Methods for predicting phenomena in the high-speed flow of gases. J. Aeronaut. Sci. **16**, 69–83 (1949).

Expository article.

Heineman, M. Theory of drag in highly rarefied gases. Communications on Appl. Math. **1**, 259–273 (1948).

Expressions are found for the drag of a body in a steady stream of highly rarefied gas, on the assumption that collisions between molecules are negligibly rare compared with collisions of the molecules with the surface, and on the two alternative hypotheses that the molecules suffer specular ("optical") reflection or diffuse reflection ("scattering"). The drag is plotted against the Mach number of the stream for several body shapes on either hypothesis. A second approximation in which molecular collisions are not wholly ignored is then given and applied to the case of a flat plate perpendicular to the stream. *M. J. Lighthill.*

Roper, Gwendolen M. The flat delta wing at incidence, at supersonic speeds, when the leading edges lie outside the Mach cone of the vertex. Quart. J. Mech. Appl. Math. **1**, 327–343 (1948).

This attack consists in transforming the linearized (Prandtl-Glauert) equation for the disturbance velocity components u, v, w into Laplace's equation for the region inside the Mach cone of the vertex and the one-dimensional wave equation outside. The transformation used is somewhat different from Chaplygin's [see Stewart, Quart. Appl. Math. **4**, 246–254 (1946); these Rev. **8**, 109]: namely, if the Cartesian coordinates are x, y, z (x being in the stream direction), u, v, w satisfy $P_{\sigma\sigma}+P_{\theta\theta}=0$, where $\operatorname{sech}\sigma=\beta(y^2+z^2)^{\frac{1}{2}}x^{-1}$ and $\tan\theta=z/y$. Here β denotes $(M^2-1)^{\frac{1}{2}}$ and M is the stream Mach number. The methods of solution for the inside region are otherwise similar to those of Stewart. Outside the Mach cone, u, v, w satisfy $P_{\chi\chi}-P_{\theta\theta}=0$, where $\sec\chi$ is now written in place of $\operatorname{sech}\sigma$. A method of Fourier expansions is used to obtain suitable solutions for this region.

From the solutions, u and v are evaluated on the wing, which lies approximately in the plane $z=0$. Lift and drag are calculated. The results for u, lift and drag are not new, and are stated to agree with those of Puckett [J. Aeronaut. Sci. **13**, 475–484 (1946); these Rev. **8**, 109] and Ward [reference unavailable to the reviewer]. The author also considers briefly the case considered by Stewart, where the leading edges are inside the Mach cone of the vertex, and obtains agreement. *W. R. Sears* (Ithaca, N. Y.).

Brown, Clinton E. Theoretical lift and drag of thin triangular wings at supersonic speeds. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **839**, 8 pp. (1946).

Nomura, Yûkiti. The theory of two plane wings. I. Sci. Rep. Tôhoku Imp. Univ., Ser. 1. **31**, 133–151 (1943).

Two plane wings of infinite span are supposed to be fixed in arbitrary position relative to one another, except that neither is intersected by the plane of the other, in a steady stream. The disturbance potentials due to these wings are written in series such as

$$\pm aU\cdot\sin\varphi\sum_{n=0}^{\infty}A_n\int_0^{\infty}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}y\xi J_{2n+1}(\xi)e^{\pm z\xi}\xi^{-1}d\xi,$$

where φ is the angle of attack of one of the wings, y and z are Cartesian coordinates, and a, U are constants. The boundary conditions lead to an approximate process for calculating the coefficients of these expansions. Formulas for force and moment are derived in terms of these, and finally the Kutta-Joukowski conditions provide formulas for the circulations, in terms of the same. The numerical calculation of the coefficients would appear to be a lengthy process in most cases. *W. R. Sears* (Ithaca, N. Y.).

Lin, Chia-Chiao. Velocity and temperature distributions in turbulent jets. Sci. Rep. Nat. Tsing Hua Univ. **4**, 419–450 (1947).

The structure of the jet is assumed to be similar at cross sections sufficiently far from the jet discharge and consequently all mean properties may be expressed in the form $x^q\varphi(y/x^p)\equiv x^q\varphi(\eta)$, where p and q are undetermined exponents. Introducing such variables into the Navier-Stokes equations simplified by the boundary layer assumptions, total differential equations result which relate the "similar part" of the stream function $F(\eta)$ with that of the stress distribution $f(\eta)$. It is found that for both the plane and axially symmetric jets the value of p is 1 and as a result the boundary layer approximation is valid only where η is small, that is, in the central portion of the jet. Similar considerations are given to the temperature distribution and the turbulent heat transfer.

In order to solve the resulting differential equations, the turbulent stress distribution $f(\eta)$ was related to the velocity distribution $F'(\eta)$ by employing Chou's tensor equation of velocity correlations [Chinese J. Phys. **4**, 1–33 (1940); these Rev. **3**, 285] and simplifying it by assuming the pressure-velocity fluctuation correlations to vanish and the viscous dissipation to be expressed in terms of an appropriate microscale. Introducing similarity representations of the mean velocities and assuming constancy of triple correlations, relations between the turbulent shear distribution and the mean velocity follow: $f(\eta)\sim -F''(\eta)$ for the plane jet and $f(\eta)\sim(F'/\eta)'$ for the axially symmetric jet. A similar tensor equation for the temperature-velocity fluctuations allows this correlation to be expressed in terms of the mean velocity and temperature distributions.

Using these results the total differential equations for the mean velocity and temperature distribution are integrated to give solutions for both plane and axially symmetric turbulent jets. For the plane jet it is also found that by choosing the local microscale λ such that $1/\lambda^2\sim U$, the local mean velocity, the same solution follows upon proper selection of the numerical constant involved. The theoretical results for the central portion of the jets are compared favorably with the experimental results of Förthmann for the plane jet and those of Ruden and of Keuthe for the axially symmetric jet. *F. E. Marble* (Pasadena, Calif.).

Reichardt, H. Impuls- und Wärmeaustausch in freier Turbulenz. Z. Angew. Math. Mech. **24**, 268–272 (1944).

Studies of momentum and heat transfer in turbulent jets and wakes are carried out on the basis of the Reynolds equations. Instead of assuming some law for the exchange coefficients, the transfer terms are expressed in terms of mean velocity and temperature distributions. It is shown that the ratio of heat transfer to momentum transfer is proportional to the ratio of temperature to velocity. The distribution of heat transfer is calculated in one case. [Reviewer's remark. The ideas and results in this paper are similar to those contained in work of S. Corrsin [unpublished report, 1943] and the reviewer [cf. the paper reviewed above, which was submitted in 1939].]
C. C. Lin (Cambridge, Mass.).

Chou, P. Y. On velocity correlations and the equations of turbulent vorticity fluctuation. Sci. Rep. Nat. Tsing Hua Univ. **5**, 52–70 (1948).

The author's summary is as follows. From the equations of turbulent vorticity fluctuation which are derived by taking the curl of the equation of velocity fluctuation, one builds the differential equations satisfied by the correlation functions between the vorticity fluctuation components. Two assumptions are then introduced into the theory. First, both the double and triple velocity correlations between two neighboring points are expansible into power series of the ratio of the vector of their relative separation over Taylor's scale of micro-turbulence, their coefficients being linear functions of the Reynolds stress. Secondly, the change of the correlation functions with respect to a rigid body translation of the two points is much smaller than that due to a relative displacement of the two points with respect to each other. As the final equation one uses the equation of turbulent energy transport and assumes, as the third postulate, that the average energy flux carried by convection through the unit surface perpendicular to a direction is directly proportional to the gradient in that direction of the rate at which work is being done by the Reynolds stress in changing the volume and shape of an element of volume, and varies inversely with the square of the decay of turbulent energy. Then one has seven more scalar equations for the six components of Reynolds stress and Taylor's scale of microturbulence besides the usual equations for mean motion for a general fully developed turbulent flow. As applications one considers the decay of energy and vorticity in isotropic turbulence and the pressure flow within a channel. Predictions in the first case have been verified by Batchelor and Townsend's experiments, while the present theory leads, in the second case, to the logarithmic law of velocity distribution and to qualitative agreements of the mean squares of velocity fluctuation components with the existing experimental data.
C. C. Lin (Cambridge, Mass.).

Batchelor, G. K., and Townsend, A. A. Decay of turbulence in the final period. Proc. Roy. Soc. London. Ser. A. **194**, 527–543 (1948).

The authors study "the last stage of decay of a motion which was formerly turbulent" or the final period of decay of a turbulent motion, which occurs when the inertia forces are negligible. Under these conditions, the instantaneous velocity distribution in the turbulence field may be solved

as an initial value problem, as was first carried out to a certain extent by Reissner [Proc. 5th Int. Congress Appl. Mech., 1938, pp. 359–361]. It is shown that homogeneous turbulence tends to an asymptotic statistical state which is independent of initial conditions. In this asymptotic state, the energy of turbulence decays according to the law $t^{-5/2}$ and the longitudinal double-velocity correlation coefficients for two points distance r apart is $e^{-r^2/8\nu t}$, where t is the time of decay measured from some proper instant. The asymptotic time-interval correlation coefficient is found to be different from unity for very large time-intervals only.

Relevant measurements have been made in the field of isotropic turbulence down stream from a grid at relatively low Reynolds numbers. The above energy decay and space correlation relations are found to be valid at distances from the grid greater than 400-mesh lengths at a mesh Reynolds number of 650. This corresponds to a Reynolds number of turbulence $R_\lambda=5$ in a field where the initial Reynolds number is about $R_\lambda=7$. Discussions are made about the beginning of the final period, and it is shown that this period can probably be observed only for relatively low mesh Reynolds numbers, unless the period of decay is greatly extended. *C. C. Lin* (Cambridge, Mass.).

Batchelor, G. K., and Townsend, A. A. A comment on F. N. Frenkiel's note "On third-order correlation and vorticity in isotropic turbulence." Quart. Appl. Math. 7, 120 (1949).

The paper appeared in the same Quart. 6, 86–90 (1948); these Rev. 9, 520.

Elasticity, Plasticity

Rivlin, R. S. Large elastic deformations of isotropic materials. IV. Further developments of the general theory. Philos. Trans. Roy. Soc. London. Ser. A. 241, 379–397 (1948).

Using general results given in a previous paper [same Trans. Ser. A. 240, 459–490 (1948); these Rev. 10, 168], the author now concentrates on an elastic body which is isotropic before strain. This means that W (the stored energy per unit unstrained volume) is a function of I_1, I_2, I_3, the three invariants of strain with respect to rotation of axes. In a compressible material the stress components are simply expressible in terms of the strain components and the partial derivatives of W with respect to I_1, I_2, I_3; in an incompressible material the terms in I_3 disappear and a hydrostatic pressure p occurs. The author criticises other workers who have assumed stress-strain relations invariant under rotation of axes, but not derivable from a function W of I_1, I_2, I_3: in particular, he examines the stress-strain relations assumed by B. R. Seth [same Trans. Ser. A. 234, 231–264 (1935)], and shows that they are consistent with a W of the stated type only if the elastic constants satisfy a special relation. Deduction of elastic laws from experiment is discussed, and it is pointed out that a simple stress-strain law may correspond to a complicated function W, and vice-versa. In terms of a general function W of the invariants the equations of motion and boundary conditions are written down for both compressible and incompressible materials. As particular cases, the simple shear of a cuboid and the torsion of a circular cylinder are discussed. [It appears to the reviewer unfortunate that a factor $\frac{1}{2}$ is deliberately omitted in the definition of shearing strain; this omission spoils the tensor character of strain and necessitates long explicit formulae where compact indicial expressions could otherwise be used.] *J. L. Synge* (Dublin).

Dean, W. R., and Mann, E. H. The change in strain energy caused by a dislocation. Proc. Cambridge Philos. Soc. 45, 131–140 (1949).

A dislocation of the type suggested by Bragg is treated. The center of dislocation (a pair of circular cylinders joined by a slit) is excluded from the calculations. [Bragg treats the case where the center of dislocation is assumed to have the energy of latent heat; same vol., 125–130 (1949).] The stress function and the displacements for two plane strain problems are given: (1) simple shear; (2) a Bragg type dislocation. The difference in the strain energy of system (1) and that of a system comprised of "(1) plus (2)" depends upon the shape of the outer boundary; for a circular outer boundary the strain energy of the combined system is less than that of the first system alone. The conclusion is drawn that a dislocation of amount s may be expected to form whenever the shear is greater than $(2s/t)\log(t/R)$, where t is the distance between the axes of the cavities and R is the common radius. *R. C. Meacham.*

Tanrikulu, Mahmut. Ordinary zero places in a body under plane stress. Rev. Fac. Sci. Univ. Istanbul (A) 13, 205–235 (1948). (English. Turkish summary)

This paper purports to deal with plane stress, in which the stress vanishes across all planes parallel to a fixed plane. However, in plane stress there are other equations of compatibility in addition to the one given by the author, and hence the surviving stress components are considerably restricted. The equations of equilibrium and compatibility which the author uses actually correspond to plane strain or to generalized plane stress. The paper is concerned with the distribution of stress lines, i.e., curves tangent at each point to a principal direction of stress. The differential equation of the stress lines is $y'/(1-y'^2)=P(x,y)/Q(x,y)$, where P and Q are simple functions of the stress components. If P and Q are both homogeneous polynomials of degree n, the origin is called an ordinary zero place of order n. The paper gives an exhaustive discussion of the case $n=1$. *J. L. Synge* (Dublin).

Ilyushin, A. A. The theory of plasticity in the case of simple loading accompanied by strain-hardening. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1207, 7 pp. (1949).

Translated from Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 11, 293–296 (1947); these Rev. 9, 120.

Ghosh, S. On the flexure of a beam whose cross-section is bounded partly by a straight line. Bull. Calcutta Math. Soc. 40, 77–82 (1948).

The author has shown [same Bull. 39, 1–14 (1947); these Rev. 9, 256] that the flexure problem for an isotropic elastic beam can be solved if the analytic function which maps the cross section of the beam on the unit circle is known. The present paper deals with the solution of the flexure problem when the cross section is bounded partly by a straight line, and forms half of an area, symmetric about this straight line, which can be mapped conformally on the unit circle. Schwarz's reflection principle for analytic functions is employed. As a simple example, the solution is given explicitly when the cross section is the semicircle $x^2+y^2\leq 1$, $y\geq 0$. *J. B. Diaz* (Providence, R. I.).

Esmeijer, W. L. On the dynamic behaviour of an elastically supported beam of infinite length, loaded by a concentrated force. Appl. Sci. Research A. 1, 151–168 (1948).

Impulsive point loading is applied to an infinite beam on an elastic foundation which supplies viscous damping as well as a reaction proportional to displacement. Laplace transforms are employed to obtain the bending moment and deflection at the point of application of the force. The cases considered in detail are the suddenly applied force which is maintained constant and several forms of impulsive loading. A brief discussion is given of the true impact problem where the force is not known. *D. C. Drucker.*

Truesdell, C. On the reliability of the membrane theory of shells of revolution. Bull. Amer. Math. Soc. 54, 994–1008 (1948).

On the basis of the several relationships developed by the author earlier [Trans. Amer. Math. Soc. 58, 96–166 (1945); these Rev. 7, 231] he now investigates the reliability of the membrane theory, and comes to the following conclusions. (1) In an open shell or in a closed dome with a flat, spherelike apex, the stress resultants computed from the equations of the membrane theory do not show a critical response to a slight perturbation of the meridian curve provided this perturbation does not effect the curvature very strongly. (2) In a closed dome with a pointed apex a slight local change of the meridian curve at the apex may produce a very great change in the membrane stress resultants throughout the shell. Hence the membrane theory is not reliable in treating problems of nonuniform load or support for pointed domes.

Both results presuppose that the boundary condition at the apex is the so-called "ring limit condition" as defined by Flügge. The author points out that to require finiteness of the strain energy at the apex, as was suggested to the author by Stoker, might be a physically more reasonable boundary condition and might possibly lead to the elimination of the unreliability of the solutions for pointed domes; however, Stoker's boundary condition brings other difficulties into the formulation of the theory.

As another possibility of obtaining a more reliable membrane theory the author discusses the abandonment of stress resultants, and reintroduction of Aron's "stress averages" as dependent variables. He shows that this, if done consistently, may lead to a good theory. However, in formulating the equations of this theory one meets the crucial difficulty of finding appropriate expressions for the "stress averages" in terms of the derivatives of the displacements of the middle surface. *P. Neményi* (Washington, D. C.).

Adadurov, R. A. Strains and deformations in a cylindrical shell stiffened by transverse membranes. Doklady Akad. Nauk SSSR (N.S.) 62, 183–186 (1948). (Russian)

The author uses the membrane (momentless) theory of shells to solve the problem of elastic equilibrium of a cylindrical shell reinforced along its length by transverse diaphragms. The latter are assumed to be rigid in their plane and flexible in the normal direction. The loading of the shell is arbitrary. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Birman, S. E. On a problem about thin-walled tubes. Doklady Akad. Nauk SSSR (N.S.) 62, 305–308 (1948). (Russian)

The note contains the formulas for stresses produced in a long hollow square elastic rod subjected to the action of torsional couples. It is assumed that the sides of the rod are in the state of plane stress. *I. S. Sokolnikoff.*

Birman, S. E. On a problem of elastic equilibrium of an infinite strip. Doklady Akad. Nauk SSSR (N.S.) 62, 187–190 (1948). (Russian)

The note contains an explicit solution of the two-dimensional problem of elastic equilibrium of an infinite strip subjected to several special types of loading along the sides of the strip. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Budiansky, Bernard, Stein, Manuel, and Gilbert, Arthur C. Buckling of a long square tube in torsion and compression. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1751, 17 pp. (1948).

Lang, H. A. Large cylindrical bending of rectangular plates. J. Appl. Mech. 15, 335–343 (1948).

The cylindrical bending of thin elastic plates is treated by considering the general relationships among the bending moment, shearing force and membrane-type force, deduced from equilibrium considerations in combination with the linear relationship between bending moment and curvature. These quantities are expanded in series, either in terms of the arc length along the cylindrical surface normal to the generators, or in terms of the projection of this on a fixed direction. By using such series expansions in combination with approximate solutions, increased accuracy of the approximate solutions is obtained. As illustrations, problems of long uniform rectangular plates are considered with the longitudinal edges pinned or clamped to fixed supports and loaded to produce cylindrical bending.

It is stated that variations in flexural rigidity may be permitted, but it appears to the reviewer that the theory would then require some modification, since equations [1] and [2] would not be correct, and this would influence the important relation [5]. *E. H. Lee* (Providence, R. I.).

Smith, C. Bassel. Effect of hyperbolic notches on the stress distribution in a wood plate. Quart. Appl. Math. 6, 452–456 (1949).

A plate of wood, which can be treated as an orthotropic material, is bounded on two sides by hyperbolic notches given by the equation $y^2/a^2-x^2/b^2=1$ and is indefinitely extended in the other direction. The x- and y-axes are parallel to the orthotropic axes of symmetry of the wood. The plate is subjected only to forces directed parallel to the x-axis. These forces are assumed to act at great distances from the y-axis and in such a way that the traction over any section perpendicular to the x-axis is statically equivalent to a single force of magnitude P. The stress function for the resultant state of plane stress satisfies a known generalization of the biharmonic equation. The stress function satisfying the appropriate boundary conditions is found as a combination of two functions of two complex variables which, because of a transformation of one of the independent variables in the generalized biharmonic equation, reduce to particularly simple forms. Calculations of the direct stress component X_x at various points of the y-axis are made for a plate of Sitka spruce for a series of ratios of the semi-axes a and b. Particular interest attaches to the ratio of the maximum stress at the point $x=0$, $y=a$, to the average stress on the section $x=0$. From the stress function for orthotropic materials that for an isotropic material is obtained and found to agree with the known form for this function [Neuber, Z. Angew. Math. Mech. 13, 439–442 (1933)]. *H. W. March* (Madison, Wis.).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Suguri, T. On the geometrical optics. Tensor **8**, 54–80 (1948). (Japanese)

Introduction to and some remarks on Carathéodory's "Geometrische Optik" [Springer, Berlin, 1937].

Durand, Émile. Une formule nouvelle pour le calcul des phénomènes de diffraction. C. R. Acad. Sci. Paris **226**, 1812–1814 (1948).

The author recommends, for the discussion of diffraction by plane opaque screens, the use of wave functions of the form

$$(1)\quad \Phi=\tfrac{1}{2}\int\int\{G(p_1,p_2)-ip_3F(p_1,p_2)\}e^{-i(p_1x_1+p_2x_2+p_3x_3)}\frac{dp_1dp_2}{-ip_3},$$

where $p_3=(k^2-p_1^2-p_2^2)^{\frac{1}{2}}$,

$$F(p_1,p_2)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int f(\xi_1,\xi_2)e^{i(p_1\xi_1+p_2\xi_2)}d\xi_1d\xi_2,$$

$$G(p_1,p_2)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int g(\xi_1,\xi_2)e^{i(p_1\xi_1+p_2\xi_2)}d\xi_1d\xi_2,$$

and $f(x_1,x_2)$, $g(x_1,x_2)$ are functions which express the boundary conditions of the problem. To justify (1), he shows that it may be converted by formal transformations into a case of Kirchhoff's formula. *E. H. Linfoot.*

Durand, Émile. Théorie électromagnétique de la diffraction par les écrans noirs. C. R. Acad. Sci. Paris **226**, 1972–1974 (1948).

The "scalar" diffraction formulae of the note reviewed above are extended to the electromagnetic theory; only minor changes are needed. *E. H. Linfoot.*

Toraldo di Francia, G. Sulla forma intrinseca della trasformazione dell'interferenza inversa. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 48–50 (1948).

Abelès, Florin. Les couches minces simples ou multiples: Travaux théoriques récents. Rev. Optique **28**, 11–31 (1949).

Abelès, Florin. Deux théorèmes relatifs à la propagation des ondes sinusoïdales dans les milieux stratifiés quelconques. C. R. Acad. Sci. Paris **227**, 899–900 (1948).

Hammad, A. The primary and secondary scattering of sunlight in a plane-stratified atmosphere of uniform composition. Astrophys. J. **108**, 338–346 (1948).

This paper, which is purely theoretical, considers primary and secondary scattering of unpolarized light in a plane-stratified atmosphere of finite optical thickness, such as that of the earth. Approximate solutions for intensity and polarization are given as functions of direction relative to that of the incident light. Numerical solutions are promised in a later paper. *W. E. K. Middleton* (Ottawa, Ont.).

Nomura, Yûkiti. Current flow in a parallel plane conductor with circular electrodes. Sci. Rep. Tôhoku Imp. Univ., Ser. 1. **30**, 363–371 (1942).

The author calculates the stationary flow of current in an infinite conductor between two parallel planes when two equal circular electrodes are placed opposite each other on the two boundary surfaces; on either electrode the potential distribution may be arbitrary. The corresponding boundary-value problem relative to the potential equation is solved by means of Fourier series and Bessel-function integrals. The coefficients of these series follow from an infinite set of linear equations. Numerical results are obtained for the case where the potentials at the electrodes are constant and equal except for the sign. The current density at the electrodes is calculated as well as the equivalent resistance of the conductor. *C. J. Bouwkamp* (Eindhoven).

Levi-Civita, Tullio. Invarianti ametrici (senza ds^2 ausiliari) ed equazioni di Maxwell per l'etere. Pont. Acad. Sci. Comment. **7**, 21–37 (1943).

The author criticizes a paper by van Dantzig [Proc. Cambridge Philos. Soc. **30**, 421–427 (1934)] in which it was claimed that Maxwell's equations in free space could be expressed in tensor form completely independent of any choice of metric. By simplifying van Dantzig's formulas, which depend on some rather involved tensor manipulations, it is shown that the nonmetric form really repeats the magnetic flux laws twice and does not include the electric flux laws except in a very restricted case. A further discussion of the electric flux laws shows that these are not in general expressible in nonmetric form. *M. C. Gray.*

Bouwkamp, C. J. On the theory of coupled antennae Philips Research Rep. **3**, 213–226 (1948).

The problem considered here is essentially the same as that discussed by King and Harrison [J. Appl. Phys. **15**, 481–495 (1944); these Rev. **6**, 222], but the results are given in somewhat different form. The present paper includes formulas of the familiar Hallén type for the self and mutual admittances of two parallel identical antennas center-driven by arbitrary voltages, and tabulates the auxiliary functions appearing in these formulas. *M. C. Gray* (Murray Hill, N. J.).

Albert, G. E., and Synge, J. L. The general problem of antenna radiation and the fundamental integral equation, with application to an antenna of revolution. I. Quart. Appl. Math. **6**, 117–131 (1948).

The authors present a general theory of radiation from a perfectly conducting antenna formed by a surface of revolution. It is assumed that there is a gap at an arbitrary location, and that the generator there may be represented by some distribution of the electric field E over the surface of revolution which is continued across the gap. Then a general integral equation for the current in the antenna is obtained in the form:

$$\int_{l_1}^{l_2}\left(k^2\psi-\frac{d^2\psi}{dzdz_0}\right)I(z)dz=2\pi ikc\int_{l_1}^{l_2}E(z)R^2(1+R'^2)^{\frac{1}{2}}\frac{1}{r}\frac{d\psi}{dr}dz,$$

where $\psi=e^{ikr}/r$, $r^2=[R(z)]^2+(z-z_0)^2$, $R=R(z)$ is the equation of the antenna surface, and $E(z)$ is the tangential component of E over the gap surface. The authors include a discussion of radiation in a finite cavity; and also of the types of antenna excitation which lead to a determinate mathematical problem. *M. C. Gray* (Murray Hill, N. J.).

Synge, J. L. The general problem of antenna radiation and the fundamental integral equation, with application to an antenna of revolution. II. Quart. Appl. Math. 6, 133–156 (1948).

This is a continuation of the paper reviewed above and contains a detailed mathematical solution of the integral equation with various simplifying assumptions. These include that wavelength≪length of gap≪radius of antenna; that the surface extension over the gap is cylindrical and that the electric field is constant over this surface; that neither arm of the antenna is a multiple of a half wavelength (that is, the formulas do not apply in the neighborhood of antiresonance). First order approximations for current and impedance are obtained, and curves are drawn showing the current distribution for various widths of gap and different locations of the gap along the antenna. Curves also show the input resistance and reactance of a thin cylindrical antenna with an infinitesimal gap at the point of quadrisection. We may note that in the first approximation the input resistance is independent of the shape of the antenna, but the input reactance includes a shape term. Only the first approximations are worked out in detail, but an alternative method which could be used for successive approximations is suggested. *M. C. Gray.*

Rydbeck, O. E. H. On the forced electro-magnetic oscillations in spherical resonators. Philos. Mag. (7) 39, 633–644 (1948).

The resonator investigated by the author consists of four concentric spherical layers separated by spheres of radii $a<c<d$. Each layer is homogeneous but the corresponding electrical constants need not be the same. Inside the space $a<r<c$, two radially directed dipoles are present at $r=b$ and $r=b_1$, while they are an angular distance θ_1 apart. Both dipoles are either of the magnetic type (current loop) or else of the electric type. The former case is studied in detail. The field of one of the dipoles is expressed in terms of a Hertzian function, which is developed in a series of spherical wave functions [for details, see Rydbeck, Trans. Chalmers Univ. Tech. Gothenburg [Chalmers Tekniska Högskolas Handlingar] no. 34 (1944); these Rev. 8, 185]. The coefficients of the various modes are complicated functions of the dimensions of the resonator system, and are expressed in terms of reflection coefficients with respect to spherical waves reflected against the inner ($r=a$) and outer ($c<r<d$) boundaries. The values of this field at the location of the second dipole immediately give the mutual impedance of the two dipoles. The general result is applied, in particular, to metallic boundaries (high-quality resonator) and dielectric boundaries. An equivalent network in the first case near the mth resonance of the nth mode is presented, the current loops being an angular distance $\theta_1=\pi$ apart.

C. J. Bouwkamp (Eindhoven).

Bremmer, H. On the theory of spherically symmetric inhomogeneous wave guides, in connection with tropospheric radio propagation and under-water acoustic propagation. Philips Research Rep. 3, 102–120 (1948).

The author compares tropospheric propagation of radio waves in the presence of a duct (denoted by "T") with the propagation of sound waves through the ocean (denoted by "O"). In both cases there is guided propagation in a spherically symmetric nonhomogeneous medium, but in "T" the guide has only one sharply defined boundary, the surface of the earth, while in "O" there are two such boundaries, the surface and bottom of the ocean. A geometrico-optical treatment is given, based on Snell's law, $r\mu \sin\tau=\text{constant}$, where μ is the index of refraction and τ is the angle between the tangent to the ray and the vertical. The difference to be emphasized is that in "T" the product $r\mu$ first decreases to a minimum value at some level $r=r_0$ as r increases from its minimum value at the surface of the earth, $r=a$; while in "O" $r\mu$ first increases to a maximum value at some level $r=r_0$ as r decreases from its maximum at $r=a$.

It is assumed that in "T" transmitter and receiver are both at the surface of the earth, while in "O" they are both on the line of maximum level r_0. Then in "T" a ray from a point source will reach the receiver by a path which may include one or more reflections from the boundary $r=a$, while in "O" the rays oscillate about the r_0 level, though they may also be reflected, either gradually or abruptly, at either boundary. The author assumes a parabolic dependence of μ on height and discusses the path lengths of the rays as a function of the number of "hops" between transmitter and receiver. Formulas for the difference in time of arrival of consecutive rays are obtained, and also for the phase differences in the arriving rays.

The paper also includes a general discussion of the theory of super-refraction from the point of view of cut-off frequencies and real modes of propagation in an arbitrary spherically symmetric waveguide. *M. C. Gray.*

Beck, Guido. Contribution to the theory of the Cherenkov effect. Physical Rev. (2) 74, 795–802 (1948).

The model is that of a point charge moving with uniform velocity v in vacuo at $t=-\infty$ and entering normally, at $t=0$, a semi-infinite ideal classical dielectric of constant specific inductive capacity ϵ. Stationary solutions for the electromagnetic field of a uniformly moving point charge are used in analogy to the method of images. The rearrangement of the virtual charges when the point charge enters the dielectric gives rise to radiation: indeed to ensure continuity of the solutions at $t=0$ a radiation field has to be added. Two cases arise according as $v<$ or $>c/\sqrt{\epsilon}$. This transition radiation is insignificant for small velocities but increases as $v\to c/\sqrt{\epsilon}$, shows a marked maximum of intensity in the forward direction when v is slightly below this critical value, and weakens considerably when v exceeds it. For $v>c/\sqrt{\epsilon}$ there is a change to the Cherenkov effect and Cherenkov radiation is obtained showing a marked maximum of intensity in the direction of a characteristic cone. The solution agrees with that of Frank and Tamm [C. R. (Doklady) Acad. Sci. URSS (N.S.) 14, 109–114 (1937)] and of Tamm [Acad. Sci. USSR. J. Physics 1, 439–454 (1939)].

C. Strachan (Aberdeen).

Agostinelli, Cataldo. Sul problema delle aurore polari (moto di un corpuscolo elettrizzato in presenza di una sfera magnetica). Soluzioni stazionarie. Pont. Acad. Sci. Comment. 7, 399–414 (1943).

The author considers a sphere of radius a magnetized symmetrically with respect to a diameter O_1O_2 (the x-axis), with additional magnetic poles located at O_1 and O_2 of moment M_0. The potential at a point P outside the sphere is then (in an obvious notation)

$$V=\tfrac{1}{2}a^{-1}M_0(r_1^{-1}-r_2^{-1})+\sum M_n r^{-n-1}Y_n^{(0)}(\cos\theta).$$

The equipotential lines in the (x, y)-plane (O_1O_2P) are given by $V=\text{constant}$, and the lines of magnetic force are the orthogonal trajectories $W=\text{constant}$, where

$$W=\tfrac{1}{2}M_0a^{-1}(\cos\theta_1-\cos\theta_2)=\sum n^{-1}M_n r^{-n}\sin\theta\, dY_n^{(0)}/d\theta.$$

The equations of motion of an electron in this field are shown to be $d\dot{x}/dt = \partial U/\partial x$, $d\dot{y}/dt = \partial U/\partial y$, $U = -\frac{1}{2}((c-\mu W)/y)^2$, where c is a constant of integration and $\mu = e/m$. The "stationary solutions" are those for which the electron moves in a circular orbit; hence they are determined by the conditions $\partial U/\partial x = \partial U/\partial y = 0$. Such solutions exist only when certain restrictions are imposed on the relative magnitudes and signs of the magnetic constants M_n and the integration constant c.

Various special cases are analyzed in detail. For instance, if only M_0 and M_1 differ from zero, then there are stationary solutions in the equatorial plane if M_0, M_1, and c all have the same sign, but in two planes symmetric with respect to the equatorial plane if M_0 is of opposite sign to M_1 and c.
M. C. Gray (Murray Hill, N. J.).

Kudryavcev, L. D. On some mathematical problems in electric circuit theory. Uspehi Matem. Nauk (N.S.) **3**, no. 4(26), 80–118 (1948). (Russian)

This is a summary of a series of seminar lectures on modern electric circuit theory. It is entirely expository. The first part deals with the topological aspect of circuits: linear graphs, orientation of branches, mesh-branch matrices, etc.; the second part with analysis and some elementary aspects of synthesis of electrical circuits with lumped elements: impedance matrix and its manipulation, duality, ideal multipole transformers, one-and-two-terminal pairs and their elementary combinations, symmetrical quadripoles, synthesis of one-terminal-pairs with only two kinds of impedance elements by canonical series, shunt and continued fraction circuits, extension to quadripoles, and some remarks on equivalence. *H. G. Baerwald* (Cleveland, Ohio).

Okada, Y. On the application of tensor calculus to electrical engineering. Tensor **8**, 1–40 (1948). (Japanese)

Following mainly the method of G. Kron [The Application of Tensors to the Analysis of Rotating Electrical Machinery, Schenectady, N. Y., 1938] the author develops the analysis of electrical machinery in terms of the tensor calculus. The author summarises a previous study of statical networks and attempts to represent the theory in invariant form. He also shows how the theory can be applied to many kinds of transducers. Then he sketches the theory discussed by Kron and points out some new problems.
A. Kawaguchi (Sapporo).

Quantum Mechanics

Slansky, Serge. Sur une définition opératorielle du changement de variables. C. R. Acad. Sci. Paris **226**, 1959–1960 (1948).

Let Q_i be the operators associated with observables of a quantum mechanical system and Q_i' the corresponding operators after a change of coordinates. The author states that the change of coordinates is allowable if and only if there exists a unitary transformation T such that $TQ_i = Q_i'T$ for all i. Then the relation between the new and old wave functions ψ' and ψ is given by $\psi' = T\psi$. In the case of the relativistically invariant Dirac theory of the electron, T is not unitary but must satisfy instead the condition $T^*\alpha_4 T = \alpha_4$, where α_4 is the fourth Dirac matrix.
O. Frink (State College, Pa.).

Costa de Beauregard, Olivier. Utilisation des projecteurs dans la théorie des grandeurs non simultanément mesurables. Ann. Physique (12) **3**, 376–391 (1948).

Two physical quantities are not simultaneously measurable with precision if the Hermitian operators corresponding to them in quantum mechanics are not permutable. To describe this situation the author makes use of a three-valued logic with truth-values meaning true, false and doubtful. He considers in some detail the proper formulation of the Heisenberg uncertainty principle in terms of this logic. In particular, he treats the case of the uncertainty principle for a coordinate and the corresponding component of momentum, and for three rectangular components of angular momentum. *O. Frink* (State College, Pa.).

Costa de Beauregard, Olivier. Sur le hachage d'une onde corpusculaire. C. R. Acad. Sci. Paris **227**, 1210–1212 (1948).

Meixner, Josef. Über den Zusammenhang der Eigenwerte derHeisenbergschen S-Matrix mit den stationären Zuständen. Z. Naturforschung **3a**, 75–78 (1948).

The author gives a new proof for the connection found by Kramers and Heisenberg [Heisenberg, same Z. **1**, 608–622 (1946); these Rev. **9**, 68] between the eigenfunctions of the continuous and those of the discontinuous spectrum of a wave mechanical eigenvalue problem. The proof is based on the well-known properties of Green's function and the completeness theorem of the eigenfunctions.
F. London (Durham, N. C.).

Hu, Ning. On the application of Heisenberg's theory of S-matrix to the problems of resonance scattering and reactions in nuclear physics. Physical Rev. (2) **74**, 131–140 (1948).

The author gives a new derivation of the general structure of the nuclear dispersion formula, with the help of Heisenberg's S-matrix method. The final result contains a number of parameters which have to be determined either empirically, or, in principle, from a knowledge of the interaction potentials of the system concerned. *A. Pais.*

Wentzel, G. Zwei Bemerkungen zur Theorie der Streumatrix. Helvetica Phys. Acta **21**, 49–58 (1948).

(I) The author points out that the Heitler integral equation cannot be applied to problems of scattering by two or more scatterers since it leads to noncausal events in the sense of Stueckelberg [same Acta **19**, 242–243 (1946)]. If the distance between the scatterers is large it is possible to set up a modified integral equation which is satisfied by the correct scattering matrix.

(II) The method of analytic continuation into the complex energy plane, when applied to the scattering matrix corresponding to the Kramers-Heisenberg dispersion formula, leads to a description of the spontaneous emission of light. *L. Hulthén* (Lund).

Groenewold, H. J. Superquantization. Nederl. Akad. Wetensch., Proc. **51**, 977–989 (1948).
Groenewold, H. J. "Superquantization." II. Nederl. Akad. Wetensch., Proc. **51**, 1091–1103 (1948).

The author discusses the known isomorphism between the quantum theory of many particles in which the configuration space is $4n$-dimensional or $(3n+1)$-dimensional and the theory of second (super) quantization. He establishes this isomorphism ab initio without direct quotation

from the work of Fock [Z. Physik 75, 622–647 (1932)]. In addition he shows how negative energy states are to be taken into account but does not discuss the divergencies in the various theories, due to these states. *A. H. Taub.*

Dirac, P. A. M. The theory of magnetic poles. Physical Rev. (2) 74, 817–830 (1948).

Previous work [Dirac, Proc. Roy. Soc. London. Ser. A. 133, 60–72 (1931)] is extended to give equations of motion for magnetic poles and charged particles interacting through the electromagnetic field. Field equations are (1) $\partial_\nu F^*_{\mu\nu}=-4\pi j_\mu$, $\partial_\nu F^\dagger_{\mu\nu}=-4\pi k_\mu$, and equations of motion

$$m(d^2z_\mu/ds^2)=e(dz^\nu/ds)F_{\mu\nu}(z)$$

and

$$m(d^2z_\mu/ds^2)=g(dz^\nu/ds)F^{\dagger*}_{\mu\nu}(z)$$

are used, where j_μ is the charge-current-density vector and k_μ its magnetic analogue, $F^\dagger_{\mu\nu}$ the dual of the antisymmetrical tensor $F_{\mu\nu}$, z_μ the coordinates of a particle, and asterisks denote slightly modified values which avoid certain singularities. By (1) the relation $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ cannot be everywhere valid and is assumed to fail on a line of points called a "string," terminating on poles or at infinity. An action principle involves among its variables "unphysical" variables describing these strings. The resulting Hamiltonian formulation is quantised by standard methods and the quantisation demands that the charges and poles are integral multiples respectively of e_0, g_0, where $e_0g_0=\hbar c/2$. Experimentally $e_0^2=\hbar c/137$: thus $g_0^2=137\hbar c/4$ which is numerically large. Thus opposite poles require great energy for separation. *C. Strachan* (Aberdeen).

Schwinger, Julian. Quantum electrodynamics. I. A covariant formulation. Physical Rev. (2) 74, 1439–1461 (1948).

This paper is the first of a series devoted to the systematic exposition of the new radiation theory developed by the author (a similar theory has been proposed independently by S. Tomonaga [Progress Theoret. Physics 1, 27–42 (1946); these Rev. 10, 226; Physical Rev. (2) 74, 224–225 (1948)]). The theory has given a satisfactory account of several experimental phenomena which were beyond the capacity of earlier theories to explain. Most earlier attempts to improve upon the Heisenberg-Pauli electrodynamics concentrated upon the removal of the divergence difficulties from the theory, and effected this removal by arbitrary subtraction prescriptions which lacked both experimental confirmation and theoretical plausibility. Schwinger avoids such mutilation of the theory, and merely reformulates it, without the addition of fundamentally new concepts, in a form which is manifestly covariant with respect to Lorentz and gauge transformations. The divergences are not eliminated, but are isolated in expressions which are unobservable and cleanly separated from finite observable effects.

The mathematical basis of the theory is only slightly different from that of the older quantum electrodynamics. The chief innovation is the introduction of a state-vector $\Psi(\sigma)$, a functional of a general 3-dimensional surface σ in space-time, representing the state of a system as determined by measurements of field-quantities on σ. Previously $\Psi(\sigma)$ was defined only for a surface whose equation, in some Lorentz frame, was $t=$constant; now it is defined for every surface which is space-like in the sense that every two points on the surface are separated by a space-like interval. Corresponding to the Schrödinger equation, there is now a functional derivative equation describing the variation of $\Psi(\sigma)$ with σ, namely

$$\text{(1)}\qquad i\hbar c(\delta\Psi(\sigma)/\delta\sigma(x))=H(x)\Psi(\sigma),$$

where $H(x)$ is an interaction energy-density of the fields at a point x of space-time, and $(\delta\Psi/\delta\sigma(x))$ is a functional derivative of Ψ with respect to small variations of σ in the neighbourhood of x. Unlike the old Schrödinger equation, (1) is manifestly covariant; the author also obtains covariant and convenient forms for his field-equations and commutation-relations. In the two final sections, covariant treatments are given for the elimination of the longitudinal part of the electromagnetic field, and for the description of collision processes. *F. J. Dyson* (Princeton, N. J.).

Feynman, Richard P. Relativistic cut-off for quantum electrodynamics. Physical Rev. (2) 74, 1430–1438 (1948).

The method suggested in an earlier paper [same vol., 939–946 (1948); these Rev. 10, 222] is used in calculating the self-energy and the scattering cross-section of the electron. It is shown that the self-energy is finite and that the numerical result agrees with those of H. A. Bethe [same Rev. (2) 72, 339–341 (1947)] and V. Weisskopf [Schwinger and Weisskopf, same Rev. (2) 73, 1272 (1947)]. The radiative correction to the electron scattering cross-section is also shown to be finite. *C. Kikuchi* (East Lansing, Mich.).

Blohincev, D. I. Field theory of extended particles. Vestnik Moskov. Univ. 1948, no. 1, 83–91 (1948). (Russian)

The author has developed independently a relativistic and divergence-free classical field theory, which is substantially identical with the theory of Peierls and McManus [H. McManus, Birmingham University thesis, 1948], and similar to the theory of R. P. Feynman [Physical Rev. (2) 74, 939–946 (1948); these Rev. 10, 222]. The theory is developed from a principle of least action, the action being the integral of a Lagrangian density L; L is derived from the customary Lagrangian by replacing the interaction term $J=\rho(P)\phi(P)$ ($\rho=$charge density, $\phi=$field strength at the point P of space-time) by a modified interaction $J'=\int\rho(P')D(P-P')\phi(P)d\Omega'$, where D is an unspecified invariant function of the 4-vector $P-P'$, and the integration is over all space-time. The change from J to J' is a change from point-particles to particles whose charge-density has a finite extension in space and time. The field equations and the equations of motion of the particles, obtained from the action principle, are integrodifferential equations; consequently in addition to physically admissible solutions they possess "nonphysical" solutions which must be eliminated by suitable boundary conditions. It is shown how, in the nonrelativistic limit, the equation of motion of a particle reduces to a correct Newtonian form, the effects of the field being fully represented by the usual radiation-damping term and by a finite addition to the inertial mass of the particle.

The author finally discusses the difficulties involved in the quantization of the theory, and of any classical theory in which a Hamiltonian formulation does not exist. [These difficulties have recently been partially overcome by Feynman, Rev. Modern Physics 20, 367–387 (1948); these Rev. 10, 224, and the paper reviewed above.] *F. J. Dyson.*

Blohincev, D. I. Field theory of extended particles. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 18, 566–574 (1948). (Russian)

This is a revised version of the paper reviewed above. The chief alteration is that the particles are now considered

in interaction with a Maxwell field instead of a scalar field. The results are not significantly changed.

F. J. Dyson (Princeton, N. J.).

Faure, Robert. Opérateurs du premier et du deuxième ordre. Rôle de l'hermiticité dans leur détermination. Hamiltonien dans le cas d'un champ électromagnétique. Intégrale première du premier ordre. Signification physique des grandeurs mesurables liées aux intégrales du premier ordre et du deuxième ordre. C. R. Acad. Sci. Paris **227**, 670–671 (1948).

Touschek, B. Note on Peng's treatment of the divergency difficulties in quantized field theories. Proc. Cambridge Philos. Soc. **44**, 301–303 (1948).

The author shows that Peng's [Proc. Roy. Soc. London. Ser. A. **186**, 119–147 (1946); these Rev. **8**, 122] method of approximation does not hold for Fermi's original theory of β-decay since in this case a continuous range of energies for the intermediate states is possible. It is also pointed out that this method of approximation does not remove the infinite self-energy of a nucleon in interaction with a charged scalar meson field.

A. H. Taub (Urbana, Ill.).

Yukawa, Hideki. On the theory of elementary particles. I. Progress Theoret. Physics **2**, 209–215 (1947).

The interaction of particles of spin $\frac{1}{2}$ and obeying Dirac's equation with fields generated by other particles is expressed in the form $V=\sum_{\lambda,\mu,\nu=0}^{3}\rho_\lambda\sigma_\mu\tau_\nu V_{\lambda\mu\nu}$, where the ρ_λ, σ_μ, τ_ν are the usual matrices for $\lambda, \mu, \nu=1, 2, 3$ and ρ_0, σ_0, τ_0 are unit matrices and the $V_{\lambda\mu\nu}$ may be functions of the variables x_μ and $p_\mu=-i\hbar\partial/\partial x_\mu$ for the particle and of the field variables. For the electromagnetic field, if the potentials A_ν are functions of the x_μ alone, then the momentum representative $(p_\mu'|A_\nu|p_\mu'')$ depends on the differences $\pi_\mu'=p_\mu'-p_\mu''$ alone. It is now supposed that this is no longer true so that the A_ν are not functions of the x_μ alone. The equation $\Box A_\nu=0$ is equivalent to the operator equation $[p_\mu, [p^\mu, A_\nu]]=0$ and Maxwell's equations for free space are (1) $[p^\mu, F_{\mu\nu}]=0$, where $-i\hbar F_{\mu\nu}=[p_\mu, A_\nu]-[p_\nu, A_\mu]$ and it is supposed that the generalised Lorentz condition $[p^\mu, A_\mu]=0$ is satisfied. This can be extended to fields for particles of zero rest-mass. The left-hand side of equation (1) is not diagonal in x-space so that the presence of source particles would be accounted for by having on the right-hand side $(x_\lambda'|s_\nu|x_\lambda'')=-e\bar{\psi}(x_\lambda'')\alpha_\nu\psi(x_\lambda')$ instead of the usual diagonal current-charge-density vector. Further development of these ideas is promised.

C. Strachan (Aberdeen).

Jauch, J. M., and Watson, K. M. Phenomenological quantum-electrodynamics. Physical Rev. (2) **74**, 950–957 (1948).

The quantization of the pure radiation field in a uniformly moving refractive medium is carried through. The Hamiltonian is shown to be expressible in the form

$$H_0=\sum_r\int[N_r(\mathbf{k})+\tfrac{1}{2}]\epsilon_k d^3k,$$

where $N_r(\mathbf{k})$ is the photon number operator and

$$\epsilon_k=\frac{\kappa v^0(\mathbf{v}\cdot\mathbf{k})+\sqrt{k^2(1+\kappa v_0^2)-\kappa(\mathbf{v}\cdot\mathbf{k})^2}}{1+\kappa v_0^2}.$$

It is shown that, for velocities greater than c/n, the above expression becomes negative. It is pointed out that the existence of negative energy photons at such velocities is related to the occurrence of Čerenkov radiation.

C. Kikuchi (East Lansing, Mich.).

Jauch, J. M., and Watson, K. M. Phenomenological quantum electrodynamics. II. Interaction of the field with charges. Physical Rev. (2) **74**, 1485–1493 (1948).

The authors extend the paper reviewed above by quantizing a phenomenological field containing charged particles. The Dirac Hamiltonian is assumed for the particles. The elimination of the longitudinal field leads to the result that the longitudinal part of the Hamiltonian contains angle- and velocity-dependent terms, in addition to the usual Coulomb term. It is shown that these additional terms account for Čerenkov radiation.

C. Kikuchi (East Lansing, Mich.).

Schönberg, Mario. Quantum theory of the point electron. I. Physical Rev. (2) **74**, 738–747 (1948).

The author develops a quantum theory of the electromagnetic field, for which the radiation field potential is taken as one half the difference between the retarded and the advanced potentials. Quantization of the field leads to the result that the radiation field potentials at any two world-points commute, in contrast to the results of the Heisenberg-Pauli theory [Z. Physik **56**, 1–61 (1929)]. This fact leads to the possibility of ferreting out from an infinite number of degrees of freedom of the radiation field just those relevant to a particular problem. The possibility of obtaining a finite nonvanishing self-energy is pointed out.

C. Kikuchi (East Lansing, Mich.).

Taub, A. H. The acceleration of the Dirac electron. Univ. Washington Publ. Math. **2**, no. 3, 41–44 (1940).

The author has elsewhere [Ann. of Math. (2) **40**, 937–947 (1939); these Rev. **1**, 95] studied the tensors associated with the Dirac equations, using the 2-component spinor formalism of Veblen. In the present paper he obtains relations between these tensors, and, in particular, calculates in terms of them the quantity $J^\nu{}_{,\mu}J^\mu$ (where J^ν is the velocity 4-vector of the electron described by the Dirac equations), which represents the acceleration of the electron in the direction of the velocity. He also derives the relativistic equations of motion.

H. S. Ruse (Leeds).

Pirenne, Jean. Le champ propre et l'interaction des particules de Dirac suivant l'electrodynamique quantique. II. Arch. Sci. Phys. Nat., Geneva (5) **29**, 121–150, 207–238, 265–300 (1947).

Previous work [same Arch. (5) **28**, 233–272 (1946); these Rev. **8**, 553] is continued. Quantum electrodynamics is given in terms of action and angle variables and the particles are electrons and positrons obeying Dirac's equation. The proper field of electron or positron is considered in detail with respect to its electric and magnetic parts and the part due to virtual transitions between states of positive and negative energy. Next the interaction between two such particles of the same or opposite signs of charge is calculated. The electron-positron system is discussed and calculations are given for the fine-structure of energy levels due to various spin and orbit interactions and also due to the probability of actual annihilation with the emission of two photons. Perturbation theory is used throughout and considerable space is given to exhibition of intermediate results in the calculations.

C. Strachan (Aberdeen).